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New treatment of the self-weight and the inertial effects of rotation for the BEM formulation of 2D anisotropic solids

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ABSTRACT

As an evident drawback for using the conventional boundary element method (BEM), an extra domain integral is present in the boundary integral equation when body-force effects are involved. For 2D anisotropic elastostatics, the extra domain integral has been exactly transformed to the boundary; however, an additional line integral intersecting the domain is involved for general cases to make the transformation. For a multiply connected region, this process is quite involving and computation-wise inefficient indeed, especially when its geometry is very complicated. In this article, a new approach is proposed to make the transformation, yet without involving extra line integrals. By this approach, the BEM's notion as a boundary solution technique is completely restored. In the end, a few benchmark problems are studied to demonstrate the veracity of formulations as well as our successful implementation in an existing BEM code.

1. Introduction

Since the 1970s′, anisotropic materials have been extensively applied for various purposes and thus, research on their engineering analysis has become an important topic. For engineering analysis, it is quite to simplify practical problems to two-dimensional cases under many circumferences. Although some theoretical principals may apply for this purpose, recourse to numerical tools, such as the finite element method (FEM) and the boundary element method (BEM), is necessary for general cases, especially when complicated boundary conditions and geometrically complex domains are treated. In contrast with the domain solution technique like the FEM, the BEM has been well recognized as an efficient numerical method for the engineering analysis. However, as a drawback for the conventional BEM analysis, body-force effects (see e.g. [5,8]) are present in the boundary integral equation (BIE) as an additional domain integral. Any attempt to directly evaluate the domain integral shall inevitably confront with the problem of domain discretization that will destroy the BEM's notion as a truly boundary solution technique.

Over the years, several approaches have been proposed to deal with the domain integral, which include special volume integration schemes (e.g. [2]), particular integral approach (e.g. [3]), dual reciprocity method (e.g. [4]), approximation of radial basis functions [11], and the exact transformation technique (e.g. [5]). Among these schemes, the most elegant approach is the exact transformation technique because no further numerical approximations are required. Besides, by exactly transforming the domain integral to the boundary, the BEM's nature of boundary discretization can be fully recovered. It had not been successful to treat body forces in 2D anisotropic bodies until Zhang et al. [12] presented the exactly transformed BIE. Due to the mathematical complexity of the associated fundamental solutions for 3D anisotropic elasticity, the success of applying this technique to three-dimensional problems was achieved very recently by Shiah [7]. Although this domain integral problem for 2D anisotropic elasticity has been resolved [12], an additional line integral appears in the transformed BIE to make the exact transformation. With reference to Fig. 1 for a multiply connected region as an example, the additional integral needs to be evaluated for all intersected segments in the domain. When the source point moves along the boundary in the collocation process, many additional line integrals need to be calculated. For that, a robust code is needed to correctly determine all intersected points on the branch-cut lines of all source points. Obviously, this process is very involving. In this article, a new approach is proposed to make the exact transformation for treating body-force effects, yet without the need to add extra line integrals. In the end, some benchmark examples are presented to demonstrate the validity as well our successful implementation in the BEM analysis for body-force effects. Truly, the present approach has completely restored the BEM's feature of boundary discretization.

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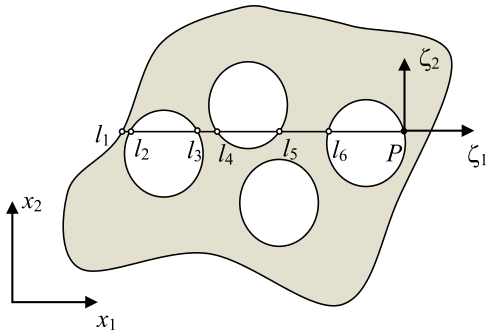

Fig. 1. The branch cut of a source point on the boundary of a multiply connected domain.

2. Transformed BIE with additional line integrals

Principally, the present work is based on the transformed BIE presented by Zhang et al. [12]. However, for the transformed BIE to hold true, additional line integrals are involved to remove the discontinuity along the branch-cut lines of source points. Our work aims to remove the necessity of including the line integrals such that the conventional BEM's notion of purely boundary discretization can be restored. Thus, a brief review of the transformed BIE is given here before further derivations for the work are presented.

For two-dimensional anisotropic elasticity, the constitutive relationship between the stress σ_{ij} and strain ε_{ij} is governed by

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{Bmatrix} = \begin{bmatrix}c_{11} & c_{12} & c_{16}\\ c_{12} & c_{22} & c_{26}\\ c_{16} & c_{26} & c_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{Bmatrix},\ \begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{Bmatrix} = \begin{bmatrix}a_{11} & a_{12} & a_{16}\\ a_{12} & a_{22} & a_{26}\\ a_{16} & a_{26} & a_{66}\end{bmatrix}\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{Bmatrix}, \tag{1}$$

where the coefficients c_{mn} and a_{mn} are the elastic stiffness and compliance constants of the material, respectively. These compliances may be given in terms of engineering constants as follows:

$$\begin{aligned}&a_{11} = 1/E_1, \quad a_{22} = 1/E_2, \quad a_{12} = -\nu_{12}/E_1 = -\nu_{21}/E_2,\\ &a_{16} = \eta_{12,1}/E_1 = \eta_{1,12}/G_{12}, \quad a_{26} = \eta_{12,2}/E_2 = \eta_{2,12}/G_{12}, \quad a_{66} = 1/G_{12},\end{aligned} \tag{2}$$

where E_k is the Young's modulus in the direction of the x_k-axis and G_{12} is the shear modulus on the x_1–x_2 plane; ν_{ij} is the Poisson's ratio, and $\eta_{i,jl}$, $\eta_{ij,l}$ are the coefficients of mutual influence of the first and second kind, respectively. Eq. (2) is also applicable to the case of plane strain, provided b_{jk} is substituted for a_{jk} by

$$b_{jk} = a_{jk} - a_{j3}a_{k3}/a_{33}, \quad (j,\ k = 1,\ 2), \tag{3}$$

where, with the index 3 referring to the x_3-axis, a_{m3} are given by

$$a_{j3} = -\nu_{j3}/E_j = -\nu_{3j}/E_3, \quad a_{33} = 1/E_3, \quad a_{63} = \eta_{12,3}/E_3 = \eta_{3,12}/G_{12}. \tag{4}$$

By introducing Airy's stress functions, Lekhnitskii [13] has shown that the characteristic equation for an anisotropic material in stable equilibrium is

$$a_{11}\mu^4 - a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - a_{26}\mu + a_{22} = 0. \tag{5}$$

It has further been shown that the roots of this characteristic equation must be complex, and are given by two distinct pairs of complex conjugates. These roots are denoted by

$$\mu_j = \alpha_j + i\,\beta_j, \quad (j = 1,\ 2), \tag{6}$$

where $i = \sqrt{-1}$ and β_j must be positive from thermodynamic considerations. For brevity, the index range of 1–2 for 2D problems will be omitted for all equations in the remaining parts. By following the above notation for material properties, the generalized variable of representing the position of a field point at (x_1, x_2) can be described by

$$z_j = (x_1 - x_{p1}) + \mu_j(x_2 - x_{p2}), \tag{7}$$

where (x_{p1},x_{p2}) are the global coordinates of the source point P. When body forces are present, the associated BIE, relating the displacements u_i and the tractions t_i, is expressed as

$$\begin{aligned}&c_{ij}(P)u_i(P)\\ &= \int_S U_{ij}^*(P,\ Q)\ t_i(Q)\ dS - \int_S T_{ij}^*(P,\ Q)\ u_i(Q)\ dS + \int_V B_i(q)\ U_{ij}^*(P,\ q)\ dV,\end{aligned} \tag{8}$$

where c_{ij} is the free term, B_i is a body-force vector, U_{ij}^* and T_{ij}^* are the fundamental solutions of displacements and tractions, respectively. As derived by Lekhnitskii [13], the fundamental solutions for an arbitrary field point Q on the boundary (or a field point q in the domain) when a unit point-load is applied in the x_j direction at P are given by

$$U_{ij}^*(P,\ Q/q) = 2\mathrm{Re}\left\{\sum_{k=1}^{2}\beta_{ik}A_{jk}\ \ln z_k\right\}, \tag{9a}$$

$$T_{1j}^*(P,\ Q) = 2n_1\mathrm{Re}\left\{\sum_{k=1}^{2}\frac{\mu_k^2 A_{jk}}{z_k}\right\} - 2n_2\mathrm{Re}\left\{\sum_{k=1}^{2}\frac{\mu_k A_{jk}}{z_k}\right\}, \tag{9b}$$

$$T_{2j}^*(P,\ Q) = -\ 2n_1\mathrm{Re}\left\{\sum_{k=1}^{2}\frac{\mu_k A_{jk}}{z_k}\right\} + 2n_2\mathrm{Re}\left\{\sum_{k=1}^{2}\frac{A_{jk}}{z_k}\right\}, \tag{9c}$$

where β_{ik} and A_{jk} are complex quantities associated with the material constants [1], and Re{·} is the operator taking the real part of complex quantities. For brevity, the relation between the source and the field point $(P,Q/q)$ in all fundamental solutions will be omitted from now on.

In elastostatics, the body force vector can be represented by the gradient of a scalar potential, namely

$$B_i = \phi_{,i}\ , \quad \phi_{,ii} = C_0, \tag{10}$$

where C_O is a constant. In elasticity, the potential function of a solid (density ρ) under rotation (angular velocity ω) about the x_3-axis is given by

$$\phi = \frac{1}{2}\rho\omega^2(x_1^2 + x_2^2). \tag{11}$$

For gravitational load along the x_i-axis, it is

$$\phi = \rho g\ x_i, \tag{12}$$

where g denotes the acceleration due to gravity. Obviously, the last term in Eq. (8) is a domain integral that the present work targets to transform. As derived by Zhang et al. [12], the domain integral (represented by V_j) is rewritten as

$$V_j = \int_V \phi_{,i}\ U_{ij}^*\ dV = \int_V\left[(\phi\ \ U_{ij}^*)_{,i} - \phi\ U_{ij,i}^*\right]dV. \tag{13}$$

As a result of applying the Gauss Divergence Theorem to the first term on the right hand side of Eq. (12), one obtains

$$V_j = \int_s \phi\ U_{ij}^* n_i\ dS - \int_V \phi\ U_{ij,i}^*\ dV. \tag{14}$$

Applying the following Green's identity

$$\int_V (\phi\ H_{j,ii}^* - H_j^*\ \phi_{,ii})\ dV = \int_S (\phi\ H_{j,i}^* - H_j^*\ \phi_{,i})\ n_i\ dS, \tag{15}$$

and enforcing

$$H_{j,ii}^* = U_{ij,i}^*, \tag{16}$$

one may immediately obtain

$$\int_V \phi\ U_{ij,i}^*\ dV = \int_s (\phi\ H_{j,i}^* - H_j^*\ \phi_{,i})\ n_i\ dS + C_0\int_V H_j^*\ dV\ . \tag{17}$$

A new function W_{ji}^* is introduced to satisfy

$$W_{ji,i}^* = H_j^*. \tag{18}$$

By us of the Green's 2nd Theorem to transform the volume integral in Eq. (17), one obtains