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Convergence analysis of Laplacian-based gradient elasticity in an isogeometric framework

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ABSTRACT

A convergence study is presented for a form of gradient elasticity where the enrichment is through the Laplacian of the strain, so that a fourth-order partial differential equation results. Isogeometric finite element analysis is used to accommodate the higher continuity required by the inclusion of strain gradients. A convergence analysis is carried out for the original system of a fourth-order partial differential equation. Both global refinement, using NURBS, and local refinement, using T-splines, have been applied. Theoretical convergence rates are recovered, except for a polynomial order of two, when the convergence rate is suboptimal, a result which also has been found for the (fourth-order) Cahn-Hilliard equation. The convergence analyses have been repeated for the case that an operator split is applied so that a set of two (one-way) coupled partial differential equations results. Differences occur with the results obtained for the original fourth-order equation, which is caused by the boundary conditions, which is the first time this effect has been substantiated.

1. Introduction

Classical continuum mechanics assumes that the solid or the structure under consideration is of a dimension that is significantly larger than its underlying microstructure, so that microstructural effects can be ignored. When the effects of microstructure become dominant – as is the case with localised shear bands in softening geomaterials [1] – classical continuum mechanics is no longer sufficient. Experiments have shown that specimens of a material with the same geometry, but different dimensions, exhibit different mechanical behaviour. This is called the size effect and has been recorded for quasi-brittle materials (concrete, rock, ceramics) [2], metals [3], composites [4] and micron-scale structures [5]. Indeed, the size effect, which has been attributed to the existence of a material microstructure, is not captured by classical continuum theories. Thus, enriching the classical continuum model with an internal length scale which is related to its material microstructure, enhances its applicability. This is the motivation behind the work of Mindlin [6] and Eringen and Suhubi [7], although earlier work along the same lines has been done by the Cosserat brothers [8]. A review and historical perspective is given in [9].

In Mindlin's theory [6], twelve independent degrees of freedom at two scales of deformation were identified: three displacement components and nine microdeformation components. Three possible assumptions that can relate the microscopic deformation gradient and the macroscopic displacement were outlined. The strain energy density can be expressed as a function of strains and second derivatives of macroscopic displacements thereby obscuring the multiscale nature of the theory [6,10,11]. This special case defines gradient elasticity. In statics, there are two additional parameters with the dimension of length which could be related to the underlying material microstructure [12,13]. A simplification is achieved when these two length scales are equal – an approach credited to Aifantis [14,15]. A proper theoretical framework was provided in [16] and [17] using the principle of minimum potential energy and principle of virtual work respectively.

The Aifantis theory modifies the classical stress-strain relation by making the stress also dependent on the Laplacian of the strain, thus resulting in a fourth-order governing partial differential equation. To solve the equation, standard C^0-continuous elements cannot be used. This is because higher order terms appear in the weak form, thus requiring the derivatives of displacements to be continuous – C^1-continuity requirement. In principle, the problem can be solved by Hermitian finite elements [18,19], mixed methods [20], meshless methods [21], penalty methods [22,23], Langrange multipliers [24] and subdivision surfaces [25]. However, all these methods have their drawbacks in terms of efficiency or implementational convenience. Thus, it remains worthwhile to explore new methods for the implementation of gradient elasticity.

An alternative approach is to use an operator split that creates two

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second-order partial differential equations. In this staggered approach [26], the solution from the first equation (classical elasticity) serves as input for the second equation which solves for the gradient-enriched variables. Since this is a set of two second order partial differential equations, it can be solved with C^0-continuous elements. It is noted that the approach suggested in Reference [26] is, strictly speaking, only applicable to an infinite body where no enforcement of boundary conditions is required [27]. Although it removes strain singularities, Skalka et al. [28] found it incapable of predicting the desired stress field around a crack in composite foams (cusp-like closure at crack tip), again pointing out issues with boundary conditions i.e. difference in boundary conditions compared with the fourth-order partial differential equation. These differences have also been pointed out in [9,29].

These restrictions have motivated Skalka et al. [28] to propose a similar strategy for Eringen's model [30], i.e. a decoupling or one-way coupling for the two second order partial differential equations. An iterative procedure was proposed for Eringen's model (also formulated by Askes and Gutiérrez [31] as implicit gradient elasticity) with the length scale replaced by a parameter increment which is chosen to be arbitrarily small. However, the choice of the number of iterations and the convergence criterion are tied to crack properties; for an arbitrary geometry, the choices seem unclear and may likely incur high computational cost. Eringen's theory is an approximation of an earlier nonlocal integral formulation [32–34]. However, it has been shown that for certain loading conditions, fully nonlocal stress-strain laws used in modelling Euler-Bernoulli elastic beams give solutions that coincide with the standard local solution, and hence do not capture size effects [35]. This can only be avoided either by combining local and nonlocal curvatures in the constitutive equation or using a gradient elastic model. [36,37].

When comparing the two solution strategies, a method which fulfils the C^1-continuity requirement is needed. Isogeometric Analysis [38] is an extension of finite element analysis where the spline-based shape functions used to approximate the geometry are used for the analysis as well. Although coined and standardised in [38], other works along the same lines exist [39,40]. The original drive behind isogeometric analysis was to integrate the design and analysis processes, which has the additional benefit of capturing the exact geometry, unlike standard finite element analysis. Moreover, it comes with the advantage of ease in achieving higher degree of continuity. This is due to the Non-Uniform Rational B-Splines (NURBS) shape functions. Isogeometric analysis has been used where higher continuity is required such as in solving the Cahn-Hilliard equation [41–43], gradient damage models [44] and also in the context of gradient elasticity [45–47,17]. In [43], the direct fourth order Cahn-Hilliard equation and a mixed formulation with coupled equations have been studied using isogeometric analysis. The study concluded that direct discretisations of higher order partial differential equations are more efficient than mixed formulations but approximations of sufficient order are required to obtain optimal convergence rates.

This work compares convergence rates for the Aifantis gradient elasticity theory with and without operator split. The paper is organised as follows: Section 2 presents the Aifantis gradient elasticity formulation including the operator-split. Section 3 starts with a brief description of NURBS and Bézier extraction in isogeometric analysis [48] before discretisation of the gradient elasticity formulation with and without operator split. In Section 4, the two discretisation approaches are compared in terms of error norms and convergence rates. T-splines are introduced in Section 5 and finally, some more examples using gradient elasticity are presented.

2. Laplacian-based gradient elasticity formulations

2.1. Aifantis' gradient elasticity formulation

The gradient elasticity theory of Aifantis [14,15] is considered herein. The theory extends the classical linear elastic constitutive relations by introducing the Laplacian of the strain as follows:

$$\sigma_{ij} = D_{ijkl}(\varepsilon_{kl} - \ell^2\varepsilon_{kl,mm}) \tag{1}$$

where $\sigma_{\rm ij}$ is the stress tensor, ε_{kl} is the strain tensor, and ℓ is a length scale parameter. D_{ijkl} is the constitutive tensor, and for an isotropic linear elastic material, it is given by:

$$D_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{kl} + \mu\delta_{il}\delta_{kl} \tag{2}$$

λ and μ are Lamé constants, and $\delta_{\rm ij}$ is the Kronecker delta. The accompanying equilibrium equations are:

$$\sigma_{ij,j} + b_i = 0 \tag{3}$$

where a comma denotes partial differentiation and b_i are the body forces. Substituting the stress-strain relation, Eq. (1), and assuming small displacement gradients, one obtains the following fourth-order partial differential equation:

$$D_{ijkl}(u_{k,jl} - \ell^2 u_{k,jlmm}) + b_i = 0 \tag{4}$$

where u_k are the displacement components.

2.2. Ru-Aifantis theorem: operator-split

In the staggered approach of the Ru-Aifantis theorem, the fourth-order equation in Eq. (4) is split into two second order partial differential equations [9,49]:

$$D_{ijkl}u^c_{k,jl} + b_i = 0 \tag{5}$$

$$u_k - \ell^2 u_{k,mm} = u^c_k \tag{6}$$

where u^c_k is the displacement field that obeys the classical elasticity equation eq. (5), hence the superscript $(\bullet)^c$. Eq. (5) is first solved for u^c_k and the result is used in Eq. (6) to solve for u_k. Thus, there is one-way coupling between them.

3. Isogeometric finite element discretisation

In traditional finite element analysis, Lagrange polynomials serve as the basis or shape functions. Isogeometric analysis replaces these Lagrange polynomials with splines which are also used in generating the geometry. This implies that both geometry and finite element analysis are based on spline functions and hence the name isogeometric analysis. NURBS or Non-Uniform Rational B-splines is the most widely used spline technology and this influenced its choice as a starting point in the seminal work where isogeometric analysis was proposed [38].

3.1. NURBS shape functions

A NURBS curve, $\mathbf{T}(\xi)$, is defined by a set of control points $\mathbf{P} = \{P_a\}_{a=1}^n \in \mathbb{R}^d$, a knot vector with increasing parametric coordinate values $\mathbf{\Xi} = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$, and a set of rational basis functions $\mathbf{R} = \{R^n_{a,p}\}_{a=1}^n$ with p being the polynomial degree, and n the number of basis functions:

$$\mathbf{T}(\xi) = \sum_{a=1}^{n} P_a R_{a,p}(\xi) \tag{7}$$

The individual coordinates of the knot vector are called knots which are analogous to nodes in standard finite elements and the interval between knots is a knot span. Unlike nodes, knots are usually not interpolatory. If the first and last knots are repeated $p + 1$ times, the knots become interpolatory, and the knot vector is said to be open. The basis functions of a NURBS curve are expressed as:

$$R_{a,p}(\xi) = \frac{w_a B_{a,p}(\xi)}{\mathbf{W}(\xi)} \tag{8}$$