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Determining the reference geometry of plastically deformed material body undergone monotonic loading and moderately large deformation

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ABSTRACT

This article presents an inverse method for predicting the reference geometry of plastically deformed material body. The reference configuration is found by solving an elastic-plastic boundary value problem to determine an inverse deformation that maps the spatial material points back to their reference positions. Rate-type elastoplastic constitutive laws are employed in the inverse analysis. When the stress exceeds the yield limit, the plastic flow is invoked and plastic variables are predicted. The ensuing stress field satisfies equilibrium and yield condition. However, the loading history is replicated only approximately and therefore the reference configuration is approximately recovered. The method is limited to a certain family of deformations. In this work, we restrict the method to problems involving monotonic loading and moderately large deformations. Numerical examples demonstrate that the method is effective and reasonably accurate for such problems.

1. Introduction

Finding the reference geometry of a finitely deforming material body is of great interest to many engineering applications. For elastic material, this problem has been well-studied. Yamada [1] and Govindjee et al [2,3] pioneered an inverse method that directly solves the equilibrium boundary value problem for the reference configuration, and this approach has led to finite element implementations that are similar to standard forward elements [2–8]. Structural inverse problems have also been investigated [9–12]. Recently, the inverse method finds applications in biomedical analysis to deal with problems for which only deformed configurations are known at the onset [13,10,14,6,15]. It was reported that, for some biological systems, the inverse method also helps to address the issue of lack of information of material properties [16,17].

Theoretically, elastoplastic deformations are history dependent and thus the inverse problem is not well-posed. The inverse solution is not unique unless the loading history or the plastic strain in the deformed state is known. Nonetheless, there has been a strong practical interest in the inverse elastoplastic problems. In the sheet metal forming community, a large body of work has been devoted to determining the initial blank geometry of workpieces based on a target final geometry using inverse analysis [18–25]. The hallmark of these works is to solve the equilibrium boundary value problem inversely to determine the initial geometry. Early developments mostly employed membrane theory to describe the mechanical behavior of thin metal sheets and implemented a one-step scheme to obtain the solution [20,19,26]. Bending effect was incorporated by using shell theory [21,25] and three dimensional constitutive laws [27]. These works mostly adopted the deformation theory of plasticity which assumes that each material point undergoes proportional loading and the axes of the strain are fixed. To better capture the history effect, multi-step schemes which admitted the proportional loading assumption stepwise were introduced [18,21,25]. Lately this inverse approach has been utilized in forging applications [28].

The inverse formulation based on the deformation theory of plasticity was found to give good strain estimation but poor stress prediction. To improve the stress estimation a pseudo-inverse approach was developed by Guo et al [29–33]. In this approach, geometrically realistic intermediate configurations were introduced. The inverse method was used to adjust the intermediate configurations succeedingly starting from the final configuration. The flow theory of plasticity and also damage theory were utilized in the inverse update process.

The contributions cited above demonstrated the usefulness of inverse analysis in forming applications. However, since the deformation theory was adopted mostly, the material was treated algorithmically as elastic. Also, certain prior information about the shape of the initial and/or intermediate configurations was incorporated in the analysis and these conditions are pertinent only to forming or forging

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applications. It remains unclear how well an elastoplastic deformation can be inversely predicted in a general setting. An attempt towards solving inverse elastoplastic problem using the algorithmic framework of [1,2] was reported in Germain et al. [34]. The authors demonstrated that when the plastic variables are known *a priori* the inverse approach can be used to recover the reference geometry. This finding underscored the fact the elastoplastic inverse problem is ill-posted. The inverse solution is not unique unless the plastic variables, or alternatively the loading path, are given. In reality, the plastic variables are unknown at the onset. To cope with this issue, the authors introduced a recursive process [35,36] consisting of iterative loops of inverse and forward analyses using the inverse computation to predict the reference geometry and the forward analysis to engage the plastic flow. In each loop, an inverse step is carried out first, with plastic variables fixed at their current value. This is followed by a forward step applied on the predicted reference geometry to generate the plastic variables. In essence, this is an operator-splitting scheme which splits the inverse problem into two sub-problems: determining the reference geometry and determining the plastic variables. The plastic flow is introduced in the forward step to recover the loading path.

In this work, we explore a direct approach of inverse analysis for elastoplastic materials governed by the flow theory of plasticity. The given information is the current geometry, applied forces and material's constitutive law. The unknowns are the reference geometry and plastic variables in the current state. We propose to use the elastoplastic constitutive law directly in the framework of [2,3]. When the stress is found to exceed the yield limit, the plastic flow is invoked and the plastic variables are predicted. The analysis will yield a reference configuration and a set of plastic variables. The ensuing stress satisfy equilibrium and yield condition. However, since the analysis starts with the current geometry, the actual strain history is not replicated and thus the analysis can only yield approximate solutions. The premise is, if the inverse "loading path" is somehow close to the actual loading history, the inverse solution is expected to be reasonably accurate. The method, therefore, is not a general approach but limited to a certain family of elastic-plastic deformations for which the strain history can be reasonably replicated in the inverse process. Here, we focus on problems involving monotonic loading and moderately large deformations. The rationale will be explained later.

The remainder of the article is organized as follows. To set the stage, the inverse elastoplastic boundary value problem is briefly described in Section 2. A finite element formulation is outlined in Section 3. The formulation utilizes existing material models and therefore only the element level computation is presented. Numerical examples are presented in Section 4.

2. Inverse elastoplastic problem

We seek to find a reference configuration of a plastically deformed material body based on the knowledge of (1) a deformed configuration of the body, (2) the applied body force, (3) boundary conditions including Cauchy traction data and displacement data, and (4) the constitutive law of the material. As alluded earlier, the approach is to determine the inverse deformation by solving the following boundary value problem: find the inverse motion $\boldsymbol{\Phi}: \Omega \mapsto \mathcal{B} \in R^3$ such that

$$\begin{aligned} &\sigma_{ij,j} + b_i = 0 && \text{in } \Omega \\ &\boldsymbol{\Phi} = \overline{\boldsymbol{\Phi}} && \text{on } \partial\Omega_u \\ &\sigma_{ij} n_j = \bar{t}_i && \text{on } \partial\Omega_t. \end{aligned} \quad (1)$$

Here $\boldsymbol{\sigma}$ is the Cauchy stress, b_i is a component of the body force per unit current volume, $\bar{t}_i$ is a component of the prescribed boundary traction, Ω is the given current configuration, and $\mathcal{B}$ is the sought reference configuration. The inverse deformation $\boldsymbol{\Phi}(\mathbf{x}, t)$ is the kinematic inverse of the forward deformation $\varphi(\mathbf{X}, t)$ at any fixed time. The gradient $\mathbf{f} := \partial_{\mathbf{x}} \boldsymbol{\Phi}(\mathbf{x}, t)$ is the inverse of the forward deformation gradient $\mathbf{F} := \partial_{\mathbf{X}} \varphi(\mathbf{X}, t)$. If, during the inverse solution process the stress is found to exceed a given elastic limit, the problem cannot be treated as elastic. An elastoplastic constitutive law is then invoked.

We focus on rate-independent plastic behavior described by the finite strain elasto-plasticity theory presented in, e.g. [37–39]. This theory treats an elastoplastic material as a family of elastic materials parameterized by plastic variables. In particular, we will utilize constitutive forms that take a metric-like plastic deformation tensor $\mathbf{C}_p$ as a primitive plastic variable [40,41]. For the model employed later in the simulation, the plastic variables include the tensor $\mathbf{C}_p$ and an equivalent plastic strain, e_p. At fixed plastic variables, the stress is given by a (hyperelastic) function of Cauchy-Green deformation tensor $\mathbf{C}$ and the plastic variables. The admissible stress lies in a (convex) region in the stress space bounded by a yield surface. If the stress tends to penetrate the yield surface, plastic flow is activated to bring the stress back to the yield surface.

In the numerical solution the elastoplastic constitutive equation is handled in essentially the same manner as in the forward analysis. Given a predicted reference configuration at time step $n + 1$, the deformation tensor is computed from

$$\mathbf{f}_{n+1} = \partial_{\mathbf{x}} \boldsymbol{\Phi}_{n+1}, \quad \mathbf{C}_{n+1} = \mathbf{f}_{n+1}^{-T} \mathbf{f}_{n+1}^{-1} \quad (2)$$

A trial stress $\mathbf{S}_{n+1}^{Tr}$ is computed using the strain $\mathbf{C}_{n+1}$ and the current plastic variables. If the trial stress satisfies the yield condition, then $\mathbf{S}_{n+1} = \mathbf{S}_{n+1}^{Tr}$ and plastic variables remain intact. Otherwise, a return mapping is performed to project the stress back and update the plastic variables. Note that the plastic tensor $\mathbf{C}_p$ is a field variable defined in the (iteratively determined) reference configuration and it predicted along with the latter. The yield condition is enforced at every step and hence the stress satisfies the yield condition at the end.

Although the treatment of elastoplastic response is algorithmically the same as in the forward analysis, there is a fundamental difference: the stain history (e.g. the loading path) is inferred from the inverse deformation. In general, the inverse strain history cannot be the same as the forward one, and thus, the actual loading path is not exactly replicated. The difference in the strain history is the root cause for the inverse solution to be approximate. A pre-requisite for the method to work is that the inverse loading path remains somehow close to the forward one. Under this circumstance, we expect that the predicted reference configuration and plastic variables are reasonably accurate. Below, we speculate some conditions for the loading paths to be close. A rigorous error analysis is beyond the scope of this work.

1. Monotonicity of loading. We first require the loading to be monotonic. To the leading order, the strain path in a monotonic loading is a line between the starting and end points in the strain space. This path depends largely on the end points, and thus is more likely to be reproduced in the inverse process. Physically, since there is no unloading or reverse plastic loading, the history influence is less prominent. Note that although the monotonicity condition seems restrictive, there is a wide range of practical problems that can fit into this category. For example, most forming or casting processes are essentially monotonic; the deformation increases continuously, bearing little or no reversal loading.
2. Moderately large deformation. Another restriction is that the deformation cannot be arbitrarily large. In elastoplastic analysis the solution is typically obtained incrementally. In the inverse approach, the predicted configuration at an intermediate step t_n is a partially recovered *reference* configuration, in contrast to a partially deformed *current* configuration in the forward analysis. The stress in the forward analysis satisfies equilibrium on the partially deformed current configuration, whereas the stress in the inverse analysis always achieves equilibrium on the given, full current configuration. The intermediate stresses are different from the equilibrium perspective. If the deformation is too large, the