Finite Elements in Analysis and Design 134 (2017) 41–54

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

Energy-momentum method for co-rotational plane beams: A comparative study of shear flexible formulations

Sophy Chhang[a,b], Jean-Marc Battini[b,*], Mohammed Hjiaj[a]

[a] Structural Engineering Research Group/LGCGM, INSA de Rennes, Université Bretagne Loire, 20 avenue des Buttes de Coësmes, CS 70839, 35708 Rennes Cedex 7, France

[b] Department of Civil and Architectural Engineering, KTH Royal Institute of Technology, SE-10044 Stockholm, Sweden

ARTICLE INFO

Keywords:
Co-rotational formulation
Energy-momentum method
Nonlinear dynamics
2D beams
Shear

ABSTRACT

This paper presents an energy-momentum method for three dynamic co-rotational formulations of shear flexible 2D beams. The classical midpoint rule is applied for both kinematic and strain quantities. Although the idea as such was developed in previous work, its realization and testing in the context of co-rotational Timoshenko 2D beam elements is done here for the first time. The main interest of the method is that the total energy and momenta are conserved. The three proposed formulations are based on the same co-rotational framework but they differ in the assumptions done to derive the local formulations. Four numerical applications are used to assess the accuracy and efficiency of each formulation. In particularly, the conservation of energy with a very large number of steps and the possibility to simplify the tangent dynamic matrix are investigated.

1. Introduction

Flexible beams are used in many applications, for instance large deployable space structures, aircrafts, wind turbines propellers and offshore platforms. These structures undergo large displacements and rotations, but still with small deformations. The simulation of their nonlinear dynamic behaviour is usually performed using beam finite elements. The co-rotational method is a very attractive approach to derive highly nonlinear beam elements [1–18]. The fundamental idea is to decompose the motion of the element into rigid body and pure deformational parts through the use of a local system which continuously rotates and translates with the element. The deformational response is captured at the level of the local reference frame, whereas the geometric non-linearity induced by the large rigid-body motion, is incorporated in the transformation matrices relating local and global quantities. The main interest is that the pure deformational parts can be assumed small and can be represented by a linear or a low order nonlinear theory [19–28].

One important issue in the co-rotational method is the choice of the local formulation. Whereas the Euler-Bernoulli beam theory is completely sufficient for the applications of slender beams, the Timoshenko beam theory takes into account shear deformation, making it suitable for describing the behaviour of short beams, composite beams, or beams subject to high-frequency excitation. The classical and simplest Timoshenko local element is obtained by using linear shape functions, a linear strain-displacement relation and a reduced integration [29–31]. However, such a formulation requires a large number of elements in order to obtain accurate results. Several alternatives for the local part are possible in order to obtain a more efficient element: a mixed approach in which the displacements and the stress are interpolated independently [32–35], an enhanced strain formulation [36–39] or the Interdependent Interpolation element (IIE) [40].

Regarding the inertia terms in the co-rotational context, Crisfield et al. [2,10] used linear local interpolations although they took local cubic interpolations to derive the elastic terms. Then, the inertia terms are easily derived and the classical constant Timoshenko mass matrix is obtained. However, Le at al. [4] adopted the IIE formulation [40], and hence cubic shape functions, to derive both the inertia and elastic terms. This leads to a formulation that requires a less number of elements but also to more complicated expressions for the inertia force vector and tangent dynamic matrix. The formulation was then extended to 3D beams without [11,12] and with [13] warping.

Another important issue in the context of non-linear dynamics is the choice of the time stepping method. In commercial finite element programs, the Alpha method [41] is usually used. However, this approach introduces numerical dissipations and consequently, the energy in the system is not conserved [42,43]. In the last decades, it has been recognized that energy conservation is a key for the stability of time-stepping algorithms in dynamics of solids and structures. Simo and Tarnow [42] were the first authors to design energy-momentum

* Corresponding author.
E-mail addresses: sophy.chhang@insa-rennes.fr (S. Chhang), jean-marc.battini@byv.kth.se (J.-M. Battini), mohammed.hjiaj@insa-rennes.fr (M. Hjiaj).

http://dx.doi.org/10.1016/j.finel.2017.04.001
Received 21 November 2016; Accepted 11 April 2017

algorithms that inherit the conservation of momenta and energy for geometrically nonlinear problem involving quadratic Green- Lagrange strains. Since then, much effort was devoted to develop energy-momentum methods for various types of formulations such as nonlinear rod dynamics [44–48], nonlinear shell dynamics [49–54], hypoelastic continuum [55,56] and elastodynamics [56–58]. With the same objective of conserving energy and momenta, Bathe [59] proposed a simple composite time stepping scheme when large deformations and long-time durations are considered.

In the co-rotational context, there have been some efforts to develop energy-momentum methods as well. Crisfield and Shi [1] proposed a mid-point energy-conserving time algorithm for two-dimensional truss elements. This concept was further developed by Galvanetto and Crisfield [3] for planar beam structures. Various end- and mid-point time integration schemes for the nonlinear dynamic analysis of 3D co-rotational beams are discussed in [10]. The authors concluded that the proposed mid-point scheme can be considered as an "approximately energy conserving algorithm". A similar approach was applied to the dynamic of co-rotational shells [16], laminated composite shells [17] and thin-shell structures [18]. Salomon et al. [14] showed the conservation of energy and momenta in the 2D and 3D analyses for the simulation of elastodynamic problems. They mentioned that, for some cases, the angular momentum is asymptotically preserved and an *a priori* estimate is obtained. However, despite of all these works, the design of an effective time integration scheme for co-rotational elements that inherently fulfils the conservation properties of energy and momenta is still an open question.

In this paper, a new energy-momentum method in the context of co-rotational shear flexible 2D beam elements is proposed. Based on the previous works of Sansour et al. [50,52], the main idea is to apply the midpoint rule not only to nodal displacements, velocities and accelerations but also to the strain fields. It means that the strains are updated by using the strain velocities instead of using directly the strain-displacement relation. The conservation of energy, linear and angular momentum is proved theoretically and also observed in the numerical applications.

Based on the same co-rotational framework, three different local formulations are implemented and tested for a large number of time steps. The respective shape functions and strain assumptions for each local formulation are presented in Table 1. The reduced integration method (RIE) is the classical Timoshenko approach based on linear interpolations and one Gauss point integration for the static terms. The mixed formulation (MX) is also based on linear interpolations but a mixed approach is used to derive the static terms. For IIE formulation, the IIE cubic shape functions [40] are used and a nonlinear shallow arch strain definition is adopted. For this last element, the expression of the tangent dynamic matrix is complicated and a possible simplification is carefully studied. For the three formulations, different predictors are tested.

The paper is organized as follows: the beam kinematics is presented in Section 2. In Section 3, Hamilton's principle and conserving properties are presented. In Section 4, the energy-momentum method is developed. The inertia and elastic terms are derived respectively in Sections 5 and 6. In Section 7, the equation of motion for all formulations is presented along with the choice of predictors and the algorithm. The proofs of the conservation of energy, linear and angular momenta are given in Section 8. In Section 9, four numerical applications are presented in order to assess the numerical performances of the proposed formulations. Finally, conclusions are presented in Section 10.

Table 1
Formulations.

Formulations	Shape function	Static term
RIE	Linear	Linear strain with reduced integration
MX	Linear	Linear strain with mixed formulation
IIE	Cubic	Shallow arch strain

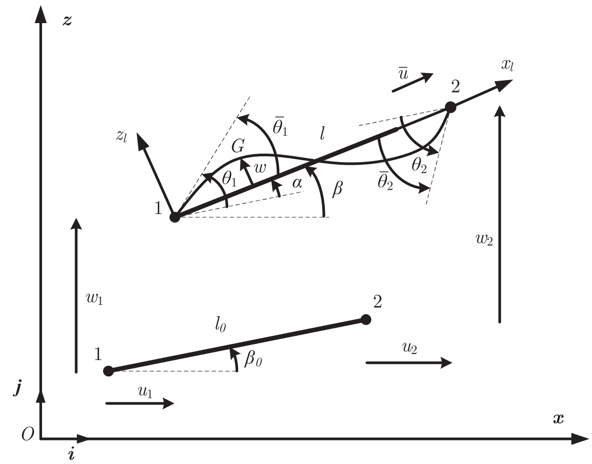

Fig. 1. Beam kinematics.

2. Beam kinematics

The kinematics of the beam and all the notations used in this section are shown in Fig. 1. The motion of the element is decomposed in two parts. In a first step, a rigid body motion is defined by the global translation (u_1,w_1) of the node 1 as well as the rigid rotation α. This rigid motion defines a local coordinate system (x_l, z_l) which continuously translates and rotates with the element. In a second step, the element deformation is defined in the local coordinate system. Assuming that the length of the element is properly selected, the deformational part of the motion is always small relative to the local coordinate systems. Consequently, the local deformations can be expressed in a simplified manner.

The vectors of global and local displacements are defined by

$$\boldsymbol{q} = [u_1 \;\; w_1 \;\; \theta_1 \;\; u_2 \;\; w_2 \;\; \theta_2]^{\mathrm{T}} \tag{1}$$

and

$$\bar{\boldsymbol{q}} = [\bar{u} \;\; \bar{\theta}_1 \;\; \bar{\theta}_2]^{\mathrm{T}} \tag{2}$$

Explicitly, the components of $\bar{\boldsymbol{q}}$ are given by

$$\begin{aligned} \bar{u} &= l - l_0 \\ \bar{\theta}_1 &= \theta_1 - \alpha = \theta_1 - \beta + \beta_0 \\ \bar{\theta}_2 &= \theta_2 - \alpha = \theta_2 - \beta + \beta_0 \end{aligned} \tag{3}$$

where l_0 and l denote the initial and current lengths of the element, respectively:

$$\begin{aligned} l_0 &= \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2} \\ l &= \sqrt{(x_2 + u_2 - x_1 - u_1)^2 + (z_2 + w_2 - z_1 - w_1)^2} \end{aligned} \tag{4}$$

The current angle of the local system with respect to the global system is denoted as β and is given by

$$\begin{aligned} c = \cos\beta &= \frac{1}{l}(x_2 + u_2 - x_1 - u_1) \\ s = \sin\beta &= \frac{1}{l}(z_2 + w_2 - z_1 - w_1) \end{aligned} \tag{5}$$

The differentiation of the expressions (3) gives

$$\delta\bar{\boldsymbol{q}} = \mathbf{B}\,\delta\boldsymbol{q} \tag{6}$$

with