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Mixed Virtual Elements for discrete fracture network simulations☆

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A B S T R A C T

The present work deals with the simulation of the flow in Discrete Fracture Networks (DFN), using the mixed formulation of the Virtual Element Method (VEM) on polygonal conforming meshes. The flexibility of the VEM in handling polygonal meshes is used to easily generate a conforming mesh even in the case of intricate DFNs. Mixed Virtual Elements of arbitrary polynomial accuracy are then used for the discretization of the velocity field. The well posedness of the resulting discrete problem is shown. Numerical results on simple problems are proposed to show convergence properties of the method with respect to known analytic solutions, whereas some tests on fairly complex networks are also reported showing its applicability and effectiveness.

1. Introduction

Effective flow simulations in underground fractured media are strategic in several practical contexts: protection of water resources, geothermal applications, Oil & Gas enhanced production and geological waste storage. All these applications share two possibly conflicting common characteristics: a high accuracy and reliability is required, whereas the uncertainty on the geometry and on the data demands for a huge number of simulations in order to provide probability distributions of the target quantities.

This work considers the problem of simulating the hydraulic head distribution in the subsoil, modeled as a Discrete Fracture Network (DFN) [1–6], which is a randomly generated set of intersecting planar polygons resembling the fractures in a surrounding porous medium. DFNs are usually characterized by enormous geometrical complexities and by the presence of a large number of fractures forming an intricate network of intersections. Many novel numerical approaches have been recently developed, in order to circumvent problems arising in efficient flow simulations in realistic DFNs. One of the main difficulties consists in the meshing process, since conventional approaches rely on the conformity of the mesh at fracture intersections in order to enforce suitable matching conditions. The generation of a mesh conforming to fracture intersections might have a high computational cost, or even fail, as a consequence of the number of geometrical constraints, and could result in poor quality triangulations for the presence of distorted elements. Furthermore, as already mentioned, input data for DFN simulations are derived from probability distribution of soil properties, thus requiring a large number of costly simulation to derive reliable statistics on the quantity of interest.

Recently, a novel code for the simulation of the flow in DFNs with stochastic input data was proposed in [7–9]. In [10,11] the complexity of DFN flow simulations is tackled resorting to dimensional reduction of the problem, removing the unknowns in the interior of the fractures and rewriting the problems at the interfaces. In [12,13] the authors use the eXtended Finite Element Method (XFEM) in order to allow for the presence of interfaces in the domain not conforming to the mesh. The XFEM is also used in [14,15]. In [16–20] the authors suggest the use of an optimization-based approach on non-conforming meshes to avoid any problem related to the generation of the mesh. The proposed optimization approach also provides a scalable resolution algorithm [21], and is used in conjunction with different discretization choices, ranging from standard finite elements, to the XFEM, [22,23], or to the new virtual element method [24]. Recently, techniques as the Mimetic Finite Difference method (MFD, [25,26]) have been used for flow

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simulations in DFNs, by [27,28], as an example, and also the new Virtual Element Method (VEM, [29–34]) was proposed, in addition to the already mentioned reference [24], also in [35,36]. In these last two works, in particular, the authors take advantage of the flexibility of virtual elements to easily generate a polygonal mesh of the fracture network that satisfies certain conformity requirements with fracture intersections.

The use of mixed formulation in DFN simulations is a widely common choice, for the possibility of a direct computation of the Darcy velocity; see among others [37,12,38–42,27,28]. This improves the accuracy for simulations in which the velocity is to be used as the transport field of an advection-diffusion process of a passive scalar, as in the case of the evolution of the concentration of a pollutant in the subsoil.

In the present work, the framework proposed in [35] is extended to the use of Mixed Virtual Elements, thus combining the reliable meshing process used therein to the mentioned advantages of the mixed formulation. The continuous advection-diffusion-reaction problem in a DFN is presented in mixed form, introducing suitable matching conditions at fracture intersections for the pressure and velocity fields. The discrete formulation with Mixed Virtual Elements of arbitrary polynomial accuracy is then derived, and a proof of well posedness is also provided. Numerical results on simple DFN configurations are first proposed, showing convergence rates of the numerical solution to the known exact solutions. Polynomial accuracy values ranging from $k = 0$ to $k = 5$ are considered. Afterwards, other numerical tests are shown on increasingly complex networks, in order to highlight the viability and the effectiveness of the method in dealing with realistic DFN configurations.

The presentation follows this outline: in Section 2 we describe the domain of interest, establish some notations and write the continuous model that describes the hydraulic head distribution within the DFN. In Section 3 the discrete formulation of the problem based on the mixed VEMs on each fracture is discussed and suitable coupling conditions at intersections are introduced. Well posedness of the discrete problem is shown. Some notes on the implementation are given in Section 4. Finally, in Section 5 validation tests are shown on advection-diffusion-reaction problems written on simple domains, together with an analysis of the performances of the method in solving pure diffusion problems on realistic DFNs.

We use the notation $\|\cdot\|_{k,\omega}$ to indicate the $\mathbf{H}^k(\omega)$-norm of vectors or scalar functions, on some set $\omega \subset \mathbb{R}^2$. In the case of a vector $\mathbf{v} = (v_1, v_2)$, we intend, e.g., $\| \mathbf{v}^2_{0,\omega} = \| \int_\omega (v_1(x,y)^2 + v_2(x,y)^2) \mathrm{d}x\mathrm{d}y$. Moreover, the symbol $[\![\mathbf{v}\cdot\mathbf{n}_\sigma]\!]_\sigma$ denotes the jump $(\mathbf{v}\cdot\mathbf{n}_\sigma^+) - (\mathbf{v}\cdot\mathbf{n}_\sigma^-)$ across a segment σ, being $\mathbf{n}_\sigma^+$, $\mathbf{n}_\sigma^-$ the unit normal vectors to σ with opposite directions. We have that $\mathbf{n}_\sigma$ is the unit normal vector to σ with one fixed orientation, and we observe that the definition of the jump is independent from the choice of $\mathbf{n}_\sigma$.

2. The continuous problem

The geometrical setting for the problem of interest is a *Discrete Fracture Network* Ω, that is a finite set of planar polygonal *fractures* intersecting in the 3D space. Each fracture in Ω is denoted by F_i, for some index $i = \{1, \ldots, N\} = \mathcal{I}$, whereas intersections between fractures are called *traces* and indicated by Γ_ℓ, for $\ell = \{1, \ldots, L\} = \mathcal{L}$. We assume, for simplicity, that each intersection occurs between exactly two fractures, and we define, for each $\ell \in \mathcal{L}$, $I_\ell = (i, j)$, with $i < j$, as the ordered couple of indices of those fractures meeting at Γ_ℓ, i.e. $\Gamma_\ell = \overline{F_i} \cap \overline{F_j}$. For each fracture F_i, $\mathcal{L}_i$ is the set of indices of those traces that F_i shares with other fractures.

The boundary of Ω, $\partial\Omega$ is split in a Dirichlet part $\Gamma_D \neq \emptyset$ and a Neumann part Γ_N with $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. Let us denote by h the hydraulic head in Ω and by h_i its restriction to F_i for $i \in \mathcal{I}$. Let further F_i be subdivided in a set of sub-domains $F_{i,j}$, $j \in \{1, \ldots, N_i\}$, such that the traces lying on F_i are now part of the boundary of some of these sub-domains. Then, the hydraulic head h in Ω is the solution of the following system of equations, which, for $i \in \mathcal{I}$ and $j \in \{1, \ldots, N_i\}$ reads as:

$$\begin{cases} \mathrm{div}(-\mathrm{K}_i \nabla h_i + \mathbf{b}_i h_i) + \gamma_i h_i = f_i & \text{in } F_{i,j}, \\ h_i = h_{Di} & \text{on } \Gamma_{Di} \cap \partial F_{i,j}, \\ (-\mathrm{K}_i \nabla h_i + \mathbf{b}_i h_i) \cdot \mathbf{n}_{\Gamma_{Ni}} = h_{Ni} & \text{on } \Gamma_{Ni} \cap \partial F_{i,j}, \end{cases} \tag{1}$$

where K_i is a uniformly positive definite tensor expressing the transmissivity of fracture F_i, whereas $\partial F_{i,j}$ is the boundary of $F_{i,j}$, and ∂F_i is the boundary of F_i which is split in a Dirichlet part $\Gamma_{Di} = \Gamma_D \cap \partial F_i$ on which the value h_{Di} is prescribed and a Neumann part $\Gamma_{Ni} = \Gamma_N \cap \partial F_i$. Across Γ_{Ni} a total (diffusive and advective) flux is imposed equal to h_{Ni}. Finally $^{\mathbf{n}}\Gamma_{Ni}$ is the outward unit normal vector to the Neumann boundary.

Problems on the fractures are coupled together by natural matching conditions expressing the continuity of h at traces and the balance of fluxes: for all $\ell \in \mathcal{L}$, if $I_\ell = (i, j)$,

$$h_i|_{\Gamma_\ell} - h_j|_{\Gamma_\ell} = 0, \tag{2}$$

$$\frac{\partial h_i}{\partial \mathbf{n}^i_{\Gamma_\ell}} + \frac{\partial h_j}{\partial \mathbf{n}^j_{\Gamma_\ell}} = 0. \tag{3}$$

where $\mathbf{n}^i_{\Gamma_\ell}$ is the unit normal vector to Γ_ℓ with a fixed orientation on F_i.

In order to introduce the variational formulation of problem (1), let us set the following functional spaces: for $i \in \mathcal{I}$ and $j = 1, \ldots, N_i$,

$$\mathrm{H}(\mathrm{div}, \mathrm{F}_{i,j}) := \left\{ \mathbf{v} \in [\mathrm{L}^2(F_{i,j})]^2 : \mathrm{div}(\mathbf{v}) \in \mathrm{L}^2(F_{i,j}) \right\},$$

$$\mathrm{H}_0(\mathrm{div}, \mathrm{F}_{i,j}) := \left\{ \mathbf{v} \in \mathrm{H}(\mathrm{div}, \mathrm{F}_{i,j}) : (\mathbf{v}\cdot\mathbf{n}\Gamma_{\mathrm{Ni}})|_{\partial F_{i,j}} = 0 \right\}.$$

We define

$$\begin{aligned}
\mathbb{V}_{i,0} &:= \{\mathbf{v}_i := (\mathbf{v}_{i,j})_{j=1,\ldots,N_i} : \mathbf{v}_{i,j} \in \mathrm{H}_0(\mathrm{div}, \mathrm{F}_{i,j}) \quad \forall \mathrm{j} \in \{1, \ldots, \mathrm{N}_i\}\}, \\
\mathbb{V}_i &:= \{\mathbf{v}_i := (\mathbf{v}_{i,j})_{j=1,\ldots,N_i} : \mathbf{v}_{i,j} \in \mathrm{H}(\mathrm{div}, \mathrm{F}_{i,j}) \quad \forall \mathrm{j} \in \{1, \ldots, \mathrm{N}_i\}\}, \\
\mathbb{V} &:= \{\mathbf{v} := (\mathbf{v}_i)_{i=1,\ldots,N} : \mathbf{v}_i \in \mathbb{V}_{i,0} \quad \forall i \in \mathcal{I}\}, \\
\mathbb{Q} &:= \{q = (q_i)_{i=1,\ldots,N} : q_i \in \mathrm{L}^2(F_i) \quad \forall i \in \mathcal{I}\}, \\
\mathbb{G} &:= \left\{\mu = (\mu_\ell)_{\ell=1,\ldots,M} : \mu_\ell \in \mathrm{H}^{\frac{1}{2}}(\Gamma_\ell) \quad \forall \ell \in \mathcal{L}\right\},
\end{aligned}$$

endowed with the following natural norms:

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbb{V}_i} &:= \left(\|\mathbf{v}\|^2_{0,F_i} + \sum_{j=1}^{N_i} \|\mathrm{div}(\mathbf{v})\|^2_{0,F_{i,j}} \right)^{\frac{1}{2}} && \forall i \in \mathcal{I}, \mathbf{v} \in \mathbb{V}, \\
\|\mathbf{v}\|_{\mathbb{V}} &:= \left(\sum_{i \in \mathcal{I}} \|\mathbf{v}\|^2_{\mathbb{V}_i} \right)^{\frac{1}{2}} && \forall \mathbf{v} \in \mathbb{V}, \\
\|q\|_{\mathbb{Q}} &:= \left(\sum_{i \in \mathcal{I}} \|q\|^2_{0,F_i} \right)^{\frac{1}{2}} && \forall q \in \mathbb{Q}, \\
\|\mu\|_{\mathbb{G}} &:= \left(\sum_{\ell \in \mathcal{L}} \|\mu_\ell\|^2_{\frac{1}{2},\Gamma_\ell} \right)^{\frac{1}{2}} && \forall \mu \in \mathbb{G},
\end{aligned}$$

in which $\| v \|_{\ell,\omega}$ denotes, as usual, the norm of function v in $\mathrm{H}^\ell(\omega)$.

By defining $\nu_i := \mathrm{K}_i^{-1}$, $\beta_i := \mathrm{K}_i^{-1}\mathbf{b}_i$, $\forall\, i \in \mathcal{I}$, and introducing, on each fracture F_i, $i \in \mathcal{I}$ the new variables $\mathbf{u}_i := -\mathrm{K}_i \nabla h_i + \mathbf{b}_i h_i$ and for each $\ell \in \mathcal{L}$ formally defining $\lambda_\ell = h|_{\Gamma_\ell}$, we can recast (1) in the following dual variational form:

Find $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_N$, *with* $\mathbf{u}_0 \in \mathbb{V}$, $h \in \mathbb{Q}$ *and* $\lambda \in \mathbb{G}$ *such that*