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Orthogonal polynomials in badly shaped polygonal elements for the Virtual Element Method☆

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ABSTRACT

In this paper we propose a modified construction for the polynomial basis on polygons used in the Virtual Element Method (VEM). This construction is alternative to the usual monomial basis used in the classical construction of the VEM and is designed in order to improve numerical stability. For badly shaped elements the construction of the projection matrices required for assembling the local coefficients of the linear system within the VEM discretization of Partial Differential Equations can result very ill conditioned. The proposed approach can be easily implemented within an existing VEM code in order to reduce the possible ill conditioning of the elemental projection matrices. Numerical results applied to an hydro-geological flow simulation that often produces very badly shaped elements show a clear improvement of the quality of the numerical solution, confirming the viability of the approach. The method can be conveniently combined with a classical implementation of the VEM and applied element-wise, thus requiring a rather moderate additional numerical cost.

1. Introduction

In the recent years a large interest on polythopal methods for PDEs has rapidly grown. In many fields of computational engineering and scientific computing the geometrical complexity is often as relevant as the model complexity. In all these situations the introduction of polyhedral or polygonal methods can introduce a great deal of flexibility that can play a relevant role in simulations.

This improved flexibility of the recently developed Virtual Element Method (VEM) has been applied in the field of geological poro-fractured media [1–5]. Geosciences very often produce applications with huge domains and terrific geometrical complexities. Within this context, the Discrete Fracture Network (DFN) model was developed for modeling the flow in geological fractured media [6–9] and is object of a very large numerical bibliography [10–20]. Due to the huge uncertainty in the definition of the underground fracture distribution, this model instantiates a fracture distribution by a stochastic procedure starting from probabilistic distributions of geometrical parameters: direction, dimension, aspect ratio; and from probabilistic distributions of thickness and other hydro-geological properties. The stochastic procedure that instantiates the fracture distribution can create geometrical complexities arbitrarily demanding for a numerical method; typically, these complexities are related, for example, to very small angles between couples of fractures, to a huge variability in the length of fracture intersections, and to disjoint fractures very close to each other [21]. Several approaches were recently applied to the DFN flow problem [10–14]. In some of these methods some geometrical simplification were required in order to construct the mesh. In [22–24,1,21,2,25–28] an optimization approach was developed in order to overcome these geometrical complexities by-passing the constraints imposed on the mesh generation process. This optimization-based approach was applied in conjunction to the classical Finite Element Method (FEM) as well as with the eXtended Finite Element Method and the VEM [1]. The VEM applied to this problem has proved a good reliability in dealing with these complexities, but, sometimes, some fracture configurations have lead to unfeasible numerical solutions [2]. A possible solution, sometimes viable, is to relax the mesh conformity requirement, resorting to a Mortar fracture matching method [3] or applying a preliminary *mesh smoothing process* [3]. Nonetheless, some very badly shaped configurations cannot be avoided, mainly on coarse meshes.

The Virtual Element Method was recently developed as a generalization of Mimetic Finite Differences, [29,30], and has been applied to a wide number of problems, such as plate bending problems [31], elasticity problems [32,33], Stokes problems [34] and the Steklov eigenvalue problem [35].

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Starting from these observations, in this paper we propose a different basis for assembling the local linear systems within the VEM, that, at a very small additional cost with respect to a classical implementation based on monomials, can largely improve the reliability of the method by limiting the condition numbers of local matrices in badly shaped elements. We remark that the proposed method aims at improving the reliability of the computations performed in the set up of the consistent part of the VEM formulation of the problem and is completely independent of the VEM stabilization that is added to the consistent part in order to get a well posed problem [36]. Moreover, our description is organized in such a way that it can be easily plugged in a standard VEM code based on scaled monomials.

In Section 3 we introduce the computation of a quasi-orthogonal polynomial basis for assembling the VEM linear system that is fully compatible with the traditional monomial basis. The two bases can be mixed on elements in the same mesh using the quasi-orthogonal basis on badly shaped elements and the traditional monomial basis on all the other elements. In Section 4 we provide a brief validation of the modified VEM construction on a general reaction-convection-diffusion problem with variable coefficients.In Section 5 we discuss the behaviour of the new basis in reducing the condition number of the projection matrices and improving the numerical solution on a simple problem. In Section 6 we compare the results provided by the classical monomial basis with the presented quasi-orthogonal basis on two critical Discrete Fracture Networks. In this Section we further discuss some simple criteria useful to determine in which elements it is beneficial to resort to the new basis and in which elements it is safe to use the monomial basis, as well as some limitations of the proposed approach.

2. Virtual element spaces

The Virtual Element Method [37,38] is a recently developed Galerkin approach to PDEs that aims at allowing the use of more generally shaped polygons than the ones allowed in the FEM context.

Consider a bounded open set $\Omega \subset \mathbb{R}^2$, partitioned by a mesh $\mathcal{T}_h$ made up of open star-shaped polygons having an arbitrary finite number of sides (even different from one polygon to another). We make the following regularity assumption: $\exists\ \gamma > 0$ such that $\forall\ E \in \mathcal{T}_h$, with diameter h_E, E is star-shaped with respect to a ball of radius larger than γh_E; more details on the regularity assumptions can be found in [36].

We define $\Pi_k^\nabla : \mathrm{H}^1_0(\Omega) \to \mathbb{P}_k(\mathcal{T}_h)$ such that, $\forall\ v \in \mathrm{H}^1_0(\Omega)$ and $\forall\ E \in \mathcal{T}_h$

$$(\nabla(v - \Pi_k^\nabla v), \nabla p)_E = 0,\ \forall\ p \in \mathbb{P}_k(E) \quad \text{and} \quad \begin{cases} (\Pi_k^\nabla v, 1)_{\partial E} = (v, 1)_{\partial E} & \text{if } k = 1, \\ (\Pi_k^\nabla v, 1)_E = (v, 1)_E & \text{if } k \geq 1, \end{cases}$$

where, $\forall\ E \in \mathcal{T}_h$, $\mathbb{P}_k(E)$ is the space of polynomials of degree up to k; its dimension is $\dim(\mathbb{P}_k(E)) = n_k = \frac{(k+1)(k+2)}{2}$.

Let us define the scaled monomials $m_{\boldsymbol{\alpha}} \in \mathcal{M}_k(E)$ up to the order k, defined as

$$\forall\ \mathbf{x} = (x, y) \in E, \quad m_{\boldsymbol{\alpha}}(x, y) := \frac{(x - x_E)^{\alpha_1}(y - y_E)^{\alpha_2}}{h_E^{\alpha_1+\alpha_2}}, \tag{1}$$

with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 \leq k$. Moreover, let $\mathcal{M}_r^*(E)$ be the set of scaled monomials of order exactly r.

Following [38,39], we introduce the local finite dimensional space

$$\begin{aligned} V_h^E := \{ & v \in \mathrm{H}^1(E) :\ \Delta \in \mathbb{P}_k(E), \\ & v \in \mathbb{P}_k(e) \quad \forall\ e \subset \partial E, \\ & \gamma_{\partial E}(v) \in C^0(\partial E) v, \\ & p_E = (\Pi_k^\nabla v, p)_E \quad \forall\ p \in \mathbb{P}_k(E)/\mathbb{P}_{k-2}(E) \} \end{aligned}$$

where the space $\mathbb{P}_k(E)/\mathbb{P}_{k-2}(E)$ is defined as $\mathcal{M}^*_{k-1}(E) \cup \mathcal{M}^*_k(E)$. We then define the global Virtual Element Space on $\mathcal{T}_h$ by gluing local spaces asking for continuity:

$$V_h := \{ v \in C^0(\Omega) \cap \mathrm{H}^1_0(\Omega) :\ v \in V_h^E \ \ \forall\ E \in \mathcal{T}_h \}.$$

The following degrees of freedom are unisolvent for V_h (see [37,38]):

1. the values at the vertices of the polygon;
2. if $k \geq 2$, for each edge $e \subset \partial E$, the value of $v \in V_h$ at $k - 1$ internal points of e. For practical purposes, we choose these points to be the internal Gauss − Lobatto quadrature nodes;
3. if $k \geq 2$, the scaled moments $\frac{1}{|E|}(v, m_{\boldsymbol{\alpha}})_E$, for all the scaled monomials $m_{\boldsymbol{\alpha}} \in \mathcal{M}_{k-2}(E)$ up to the order $k - 2$.

The above degrees of freedom are enough to build projection matrices in order to obtain local polynomial orthogonal projections from V_h to $\mathbb{P}_k(\mathcal{T}_h)$, see [40].

2.1. Example: VEM for advection-diffusion-reaction equations

Following [38], we consider the general second order problem

$$\begin{cases} -\nabla\cdot(\mu\nabla u) + \beta\cdot\nabla u + \gamma u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

whose variational formulation reads

$$(\mu\nabla u, \nabla v) + (\beta\cdot\nabla u, v) + (\gamma u, v) = (f, v). \tag{2}$$

The VEM discretization of (2) consists in defining a discrete counterpart of the bilinear form which is computable from the VEM degrees of freedom. Let

$$\begin{aligned} a_h(u_h, v_h) &:= (\mu\Pi_0^{k-1}\nabla u_h, \Pi_0^{k-1}\nabla v_h) + S((I - \Pi_\nabla^k)u_h, (I - \Pi_\nabla^k)), \\ b_h(u_h, v_h) &:= (\beta\cdot\Pi_0^{k-1}\nabla u_h, \Pi_0^{k-1} v_h), \\ c_h(u_h, v_h) &:= (\Pi_0^{k-1} u_h, \Pi_0^{k-1} v_h), \\ \mathcal{B}_h(u_h, v_h) &:= a_h(u_h, v_h) + b_h(u_h, v_h) + c_h(u_h, v_h), \end{aligned} \tag{3}$$

where S is the VEM stabilization [37,36] such that

$$\begin{aligned} & \exists\ c_*, c^* > 0 :\ \forall\ v_h \in \ker(\Pi_\nabla^k), \\ & c_* \|\nabla v_h\|^2 \leq S(v_h, v_h) \leq c^* \|\nabla v_h\|^2, \end{aligned}$$

and all the other terms of the operator $\mathcal{B}_h(.,.)$ provide the consistent part of the operator. Within these terms, the operator Π_0^{k-1} is the elementwise $\mathrm{L}^2(E)$ projection on $\mathbb{P}_{k-1}(E)$, for any $E \in \mathcal{T}_h$. For the ease of notation, we will use the same symbol also for the application of the projection operator to vectors, such as gradients, meaning a component-wise application.

Using the above definitions, we define the discrete VEM solution as the function $u_h \in V_h$ satisfying

$$\mathcal{B}_h(u_h, v_h) = (f, \Pi_0^{k-1} v_h) \quad \forall\ v_h \in V_h\ .$$

This problem is well-posed and satisfies optimal a priori error estimates [38]. In the following we focus on the construction of the local projection matrices and the local matrices and vectors required for the set up of the global discrete problem.

In the presentation given here we have considered the minimal requirement in the projections in order to preserve the expected polynomial rate of convergence (k in the energy norm) of the numerical solution [38,4].

3. Orthogonal polynomials on the generic element

All the computations performed in order to set up the VEM linear system providing the solution are based on operations between polynomial functions representing the projection of functions appearing in the consistent part of the operator and in the right-hand-side. A key issue in performing all the computations is a suitable basis for the polynomial spaces on general polygonal elements. Among the several possible options the classical and more simple choice is the scaled