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Direct computation of shakedown loads via incremental elastoplastic analysis

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ABSTRACT

The problem of shakedown analysis is considered. The mathematical programming formulations of limit and incremental elastoplastic analysis are first briefly reviewed and a convenient standard form for shakedown analysis is then suggested. This standard form can formally be viewed as a problem of limit analysis. In this way, two different solution approaches are applicable: either the problem can be solved directly using general optimization methods or the problem can be converted into an equivalent fictitious incremental elastoplastic problem and solved as such. We further show that this result holds in general for arbitrary convex mathematical programs. Thus, all the methods and techniques developed for elastoplasticity are in principle applicable to general convex programming. For the solution of shakedown problems we employ a version of the well-known implicit solution procedure in combination with a number of equally well-known techniques from general optimization theory. The resulting algorithm enables an efficient and robust treatment of cone-shaped yield constraints of the Drucker–Prager type.

1. Introduction

Shakedown analysis is well-established as a tool for assessing the safety factor against instantaneous plastic collapse, fatigue and excessive accumulated strains in structures subjected to cyclic loading [1–4]. Typically, the task is to find the factor by which a given set of cyclic loads can be magnified without the structure suffering any of the above mentioned types of failure. This factor can be assessed using either the static (lower bound) or the kinematic (upper bound) shakedown theorem. When formulated numerically the shakedown theorems generate large, nonlinear optimization problems that can be solved using general optimization methods [5–10]. Alternatively, more specialized procedures applicable only to the problem of shakedown analysis, and often also only to a limited class of yield criteria, can be employed. Examples of these latter methods include the so–called linear matching method of Ponter et al. [11,12] and the path-following procedure proposed by Casciaro and Garcea [13] and Garcea [14].

With the efficiency that general purpose optimization methods have attained over the last few decades these would seem more attractive than the latter type of methods. However, despite very significant advances in the field of applied optimization, there are still a number of outstanding issues. Indeed, although problems containing hundreds of thousands of variables are now routinely solved in many fields of engineering, science, and economics, the current state-of-the-art methods are still suffer from a certain degree of problem dependence and often require 'tuning' for a given application. This tuning includes rules for initializing and updating key algorithmic parameters and is to a large extent carried out on the basis of experience. Therefore, in the continued application of a given algorithm to different classes of problems, or even to different problems within the same class, one will eventually encounter difficulties with convergence. More seriously, however, is it often not obvious how to improve the robustness of a given optimization algorithm, even at the expense of efficiency, and the most common remedy for convergence difficulties is to adjust one or more algorithmic parameters and rerun the problem. Thus, although many modern optimization algorithms are indeed powerful tools, their use as 'black boxes' is still somewhat problematic.

This situation is in some sense analogous to the one in nonlinear finite element analysis, where it is also well-known that convergence difficulties may occur. Using appropriate physical insight, these difficulties can frequently be alleviated in an intuitively rational manner. In elastoplasticity, for example, the obvious remedy to convergence difficulties is to reduce the magnitude of the load step. Although it is hard to produce a rigorous mathematical proof that this does indeed improve the likelihood of convergence, the approach is well-established in practice and for most problems actually works.

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It is well-known that the ultimate load of an elastoplastic structure discretized by conventional displacement finite elements can be found in two ways: either by performing a step-by-step elastoplastic analysis up to the point of incipient collapse or, more directly, by maximizing the magnitude of the external loads subject to discrete equilibrium and yield constraints. There are many advantages of the latter approach, but also the significant drawback that potential convergence difficulties may be hard to address in a rational and systematic manner. On the other hand, if the former method of solution is chosen, convergence is virtually guaranteed. That is, very small load steps, local substepping, line searches, and so on, may be required, but eventually the point of collapse will be reached.

These facts have motivated a closer look at the links between convex programming and certain problems of nonlinear finite element analysis, in particular incremental elastoplastic analysis. More precisely, we pose the question: can the step-by-step solution procedure commonly used for elastoplasticity be generalized to arbitrary convex optimization problems? As will be shown in the following, the answer to this question is affirmative. To demonstrate the point we consider the problem of shakedown analysis where there is no physical incremental counterpart as in the case of limit analysis/elastoplasticity. The resulting algorithm is in many ways similar to the one of Casciaro and Garcea [13] and Garcea [14] although there are also some notable differences, not least concerning the generality of the overall approach.

The paper is organized as follows. In Section 2 we give a brief summary of the governing equations and their discretization and a relevant mathematical program is formulated. It is then shown in Section 3 that *all convex programming problems* can be cast in a form which is identical to the one commonly used for rigid-plastic limit analysis. By way of this insight two different solution approaches suggest themselves naturally: either the problem can be solved directly using general optimization methods, or the problem can be solved in a step-by-step fashion as in done in elastoplasticity. That is, instead of solving a given convex program directly, a certain amount of 'elasticity' can be added and the problem can then be solved by tracing the fictitious load–displacement curve until 'collapse', i.e. until no further increase in the objective function is possible. As such, any of the numerous methods traditionally employed for elastoplasticity are applicable. Similarly, all the procedures developed to increase robustness and/or efficiency, such as automatic load step selection, can be applied in an already manner. Next, in Section 4, a convenient standard form of shakedown analysis is given and the corresponding fictitious incremental elastoplastic problem is derived. We here emphasize that this latter problem is a standard single-surface problem, in contrast to the multi-surface formulation employed by Casciaro and Garcea [13] and Garcea [14]. Also, the formulation is quite general and not restricted to any particular type of yield criterion. However, following the trend in optimization of designing efficient algorithms for more specialized problems, we deal specifically with linear-quadratic yield constraints of the Drucker–Prager type in Section 5. In Section 6 the solution algorithm is presented. This algorithm is essentially an adaptation of the well-known implicit procedure of Simo and Taylor [15,16]. Some implementation issues particular to shakedown analysis are discussed. Finally, in Section 7 a number of test examples are solved and comparisons are made with a conventional state-of-the-art conic programming optimizer before conclusions are drawn in Section 8.

2. Shakedown analysis

In the following, the static theorem of shakedown analysis is briefly reviewed and then used as a basis for the finite element discretization.

Consider a polyhedral load domain defined by L vertices. The elastic stresses corresponding to each of these vertices are denoted $\boldsymbol{\chi}_k(\boldsymbol{x}), k=1,\dots,L$. All possible modes of plastic failure are then prevented if there exists a residual stress field $\boldsymbol{\rho}(\boldsymbol{x})$ such that

$$\begin{aligned} &\nabla^{\mathsf{T}}\boldsymbol{\rho}(\boldsymbol{x})=\mathbf{0} \quad \forall \boldsymbol{x}\in\Omega \\ &\boldsymbol{n}^{\mathsf{T}}\boldsymbol{\rho}(\boldsymbol{x})=\mathbf{0} \quad \forall \boldsymbol{x}\in\Gamma_\rho \\ &f(\boldsymbol{\rho}(\boldsymbol{x})+\alpha\boldsymbol{\chi}_k(\boldsymbol{x}),\boldsymbol{x})\le 0 \quad \forall(\boldsymbol{x},k)\in\Omega\times\mathcal{L} \end{aligned} \tag{1}$$

where $\boldsymbol{x}$ is the spatial coordinate, Ω is the spatial domain under consideration, Γ_ρ is the unsupported part of the boundary, and $\mathcal{L}=(1,L)$ is the load domain. We assume an elastic-perfectly plastic behavior described by the yield function f and an associated flow rule. In (1), ∇ is a matrix of linear differential operators. For a two dimensional continuum ∇^{T} is given by

$$\nabla^{\mathsf{T}}=\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}$$

If the above conditions are fulfilled exactly and the elastic stresses are the exact ones, the multiplier α will be a lower bound to the true elastic shakedown multiplier. Thus, we seek to maximize α subject to the above constraints.

2.1. Finite element discretization

The equilibrium constraints can be enforced weakly as

$$\int_\Omega \boldsymbol{u}^{\mathsf{T}}\nabla^{\mathsf{T}}\boldsymbol{\rho}\,\mathrm{d}\Omega=0 \tag{2}$$

where $\boldsymbol{u}$ are arbitrary (virtual) displacement fields fulfilling the kinematic boundary conditions. Integration by parts gives

$$\int_\Omega \boldsymbol{\rho}^{\mathsf{T}}\nabla\boldsymbol{u}\,\mathrm{d}\Omega-\underbrace{\int_{\Gamma_\rho}\boldsymbol{u}^{\mathsf{T}}\boldsymbol{n}^{\mathsf{T}}\boldsymbol{\rho}\,\mathrm{d}\Gamma}_{=0\text{ by }(1)_2}=0 \tag{3}$$

The residual stress and displacement fields are approximated by

$$\boldsymbol{\rho}\approx\boldsymbol{N}_\sigma(\boldsymbol{x})\boldsymbol{\rho}^h, \quad \boldsymbol{u}\approx\boldsymbol{N}_u(\boldsymbol{x})\boldsymbol{u}^h \tag{4}$$

where $\boldsymbol{N}_\sigma(\boldsymbol{x})/\boldsymbol{N}_u(\boldsymbol{x})$ contains the stress/displacement interpolation functions, $\boldsymbol{\rho}_h$ is the nodal stress vector and $\boldsymbol{u}^h$ the nodal displacements vector. The discrete weak equilibrium condition reads

$$\boldsymbol{B}^{\mathsf{T}}\boldsymbol{\rho}=\mathbf{0}, \quad \boldsymbol{B}^{\mathsf{T}}=\int_\Omega \boldsymbol{N}_u^{\mathsf{T}}\nabla^{\mathsf{T}}\boldsymbol{N}_\sigma\,\mathrm{d}\Omega \tag{5}$$

where superscripts h have been dropped so that $\boldsymbol{\rho}$ are the discrete nodal stresses.

2.2. Mathematical program

With the equilibrium constraints discretized a fully discrete mathematical programming formulation is obtained by enforcing the yield conditions at a finite number of points, typically the stress nodes, so that the task is to solve

$$\begin{aligned} &\text{maximize} \quad \alpha \\ &\text{subject to} \quad \boldsymbol{B}^{\mathsf{T}}\boldsymbol{\rho}=\mathbf{0} \\ &\qquad f_j(\boldsymbol{\rho}_j+\alpha\boldsymbol{\chi}_{j,k})\le 0 \quad \forall(j,k)\in\mathcal{S}\times\mathcal{L} \end{aligned} \tag{6}$$

where $\mathcal{S}=(1,S)$ with S being the total number of stress points (Gauss points) so that $\boldsymbol{\rho}=(\boldsymbol{\rho}_1,\dots,\boldsymbol{\rho}_S)^{\mathsf{T}}$ with each subvector $\boldsymbol{\rho}_I$ being given by $\boldsymbol{\rho}_I=(\rho_x,\rho_y,\dots,\rho_{yz})^{\mathsf{T}}$.