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Compatible diagonal-norm staggered and upwind SBP operators

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ABSTRACT

The main motivation with the present study is to achieve a provably stable high-order accurate finite difference discretisation of linear first-order hyperbolic problems on a staggered grid. The use of a staggered grid makes it non-trivial to discretise advective terms. To overcome this difficulty we discretise the advective terms using upwind Summation-By-Parts (SBP) operators, while the remaining terms are discretised using staggered SBP operators. The upwind and staggered SBP operators (for each order of accuracy) are compatible, here meaning that they are based on the same diagonal norms, allowing for energy estimates to be formulated. The boundary conditions are imposed using a penalty (SAT) technique, to guarantee linear stability. The resulting SBP-SAT approximations lead to fully explicit ODE systems. The accuracy and stability properties are demonstrated for linear hyperbolic problems in 1D, and for the 2D linearised Euler equations with constant background flow. The newly derived upwind and staggered SBP operators lead to significantly more accurate numerical approximations, compared with the exclusive usage of (previously derived) central-difference first derivative SBP operators.

1. Introduction

It is well known that higher order methods (as compared to first- and second-order accurate methods) capture transient phenomena more efficiently since they allow a considerable reduction in the degrees of freedom, for a given error tolerance. In particular, high-order finite difference methods (HOFDM) are ideally suited for problems of this type. (See the pioneering paper by Kreiss and Oliger [1] concerning hyperbolic problems.) The major difficulty with HOFDM is to obtain a stable boundary treatment, something that has received considerable past attention concerning hyperbolic and parabolic problems. (For examples, see [2–7].) For long-time simulations, it is imperative to use finite difference approximations that do not allow growth in time—a property termed "strict stability" [8].

Strictly stable HOFDM on bounded domains usually have very different resolving capabilities in the large interior domain (away from boundaries), compared to the vicinity of the boundaries. Typically, the interior accuracy is much higher compared to the boundary accuracy. Roughly speaking, for hyperbolic problems the numerical errors are caused by: 1) the time integrator, 2) the boundary treatment, and 3) the interior dispersion error from long-range wave propagation. The time integration error is typically negligible when using a high-order Runge–Kutta method, compared to the spatial errors, i.e., points 2 and 3 above. For problems with very large computational domains (compared to the wavelengths) the interior

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dispersion error is often the dominating error source. For problems with (internal and external) boundaries, and smaller domains (compared to the wavelengths), the boundary accuracy contributes significantly to the overall accuracy.

For linear hyperbolic wave-equations (an important special case of hyperbolic problems), such as Maxwell's equations, the acoustic wave equation and the elastic wave equation, there are essentially two different HOFDM approaches. The first approach is to start from first order hyperbolic form and employ either a staggered grid finite difference approximation (SGFDA) [9–11], or a regular (i.e., equidistant collocated) grid finite difference approximation (RGFDA) [6,12]. The second approach is to rewrite the continuous equations into second order hyperbolic form and employ a narrow-stencil second derivative finite difference approximation. (For examples, see [13–19].)

The first approach, on a regular grid, has two obvious disadvantages for hyperbolic wave-equations, compared to the second approach: 1) the number of unknowns increase significantly in 3D, and 2) spurious oscillations due to unresolved features might be introduced when employing a central-difference first derivative approximation on a regular grid. The stability analysis for the second approach is less mature, although some important results have been reported during the last decade [20,16,21,17–19]. The second approach is however limited to wave equations and not applicable to more general hyperbolic equations with advective terms such as the linearised Euler equations.

For the first approach, we distinguish between SGFDA and RGFDA. It is well-known [22] that SGFDA have a much smaller interior dispersion error compared to RGFDA. Hence, SGFDA is potentially the most accurate finite-difference methodology for more general first-order hyperbolic equations, including the linearised Euler equations.

For linear hyperbolic problems, there are however some challenges with SGFDA: 1) non-cartesian grids, 2) boundary conditions, and 3) problems with background flow (e.g., the linearised Euler equations in a varying atmosphere [23]). The main focus (and novelty) in the present study is to develop a SGFDA methodology that can handle the last two challenges above, in a provably stable way up to 8*th* order accuracy.

Staggered grids are also popular when solving the incompressible Navier–Stokes equations (NSE) [24–27]. The convective nonlinear terms in NSE introduce another complication (added to the above list of challenges) since the unknowns and fluxes are not located in the same positions, requiring high-order interpolation. The introduction of general curvilinear grids (related to the first challenge in the above list) will also require the introduction of high-order interpolation. In a coming study we hope to include also the first challenge (curvilinear grids) and further extend the SGFDA methodology towards nonlinear problems, such as NSE. This is however out of scope in the present study.

A robust and well-proven HOFDM for well-posed initial boundary value problems (IBVP), is to combine summation-by-parts (SBP) operators [28–31] and either the simultaneous approximation term (SAT) method [32], or the projection method [33,34,15,23] to impose boundary conditions (BC). Recent examples of the SBP-SAT approach can be found in [35–42]. The SBP operators found in literature (see for example [28–30,16,43–45]) are essentially central finite difference stencils, defined on regular grids, closed at the boundaries with a careful choice of one-sided difference stencils, to mimic the underlying integration-by-parts formula in a discrete norm. Central-difference SBP operators defined on a regular (collocated) grid will here be referred to as *traditional* SBP operators. In [46] a spectral method on arbitrary grids with SBP properties was presented.

SBP operators defined on a regular grid with non-central finite difference stencils in the interior were introduced in [42], referred to as *upwind* SBP operators. Two benefits with upwind SBP operators, compared to traditional SBP operators are: 1) the "built in damping" of spurious oscillations, and 2) favourable convergence properties (see [42] for details). However, the internal dispersion properties for upwind SBP operators are still less favourable compared to SGFDA. (See for example Fig. 1.)

SBP operators are sometimes categorized (see for example [29,30]) by the structure of their norm: a) diagonal, b) diagonal interior with block boundary closures, c) fully banded (Padé type [47]). For stability reasons (see for example [48–52]) the diagonal-norm SBP operators are the preferred choice in practice. The diagonal-norm SBP operators presented in literature have traditionally been defined on non-staggered (collocated) grids. However, in a recent study [53] diagonal-norm first derivative SBP operators on staggered grids are introduced, referred to as *staggered* SBP operators. In [53] the staggered SBP operators are combined with the SAT technique to solve 2D wave equations on staggered grids. Staggered SBP operators have recently [54] also been developed for contending with coordinate singularities in axisymmetric wave propagation with strongly enforced BC (sometimes refereed to as the injection method).

To achieve a staggered grid discretisation with the SBP-SAT methodology, applicable to more general hyperbolic systems (including advective terms) requires the combination of (collocated) upwind (or traditional) SBP operators and staggered SBP operators using the same norms. Such operators have not yet been presented in literature. Our first main goal is the construction of novel upwind and staggered SBP operators in the same diagonal norms. The second main goal is the formulation of strictly stable SGFDA of the 2D linearised Euler equations with background flow (wind), using the novel SBP operators in combination with the SAT boundary treatment.

In Section 2 the SBP-SAT method is introduced in 1D. The stability analysis for a 1D hyperbolic system, when combining upwind and staggered SBP operators is discussed in Section 3. In Section 4 the accuracy and stability properties of the newly developed SBP operators are verified by performing 1D numerical simulations. The stability analysis for the 2D Euler equations employing this novel combination of upwind and staggered SBP operators is discussed in Section 5. Verification of accuracy and stability by numerical studies of the 2D Euler equations is performed in Section 6. Section 7 summarizes the work. The coefficients for the novel SBP operators are included online as supplementary data for this paper, as described in Appendix B.