Journal of Computational Physics 350 (2017) 326–342

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

On the nonoscillatory phase function for Legendre's differential equation

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A R T I C L E   I N F O

Article history:
Received 4 February 2017
Received in revised form 3 August 2017
Accepted 20 August 2017
Available online 31 August 2017

Keywords:
Special functions
Fast algorithms
Nonoscillatory phase functions

A B S T R A C T

We express a certain complex-valued solution of Legendre's differential equation as the product of an oscillatory exponential function and an integral involving only nonoscillatory elementary functions. By calculating the logarithmic derivative of this solution, we show that Legendre's differential equation admits a nonoscillatory phase function. Moreover, we derive from our expression an asymptotic expansion useful for evaluating Legendre functions of the first and second kinds of large orders, as well as the derivative of the nonoscillatory phase function. Our asymptotic expansion is not as efficient as the well-known uniform asymptotic expansion of Olver; however, unlike Olver's expansion, it coefficients can be easily obtained. Numerical experiments demonstrating the properties of our asymptotic expansion are presented.

1. Introduction

The Legendre functions of degree $\nu \in \mathbb{C}$ — that is, the solutions of the second order linear ordinary differential equation

$$y''(z) - \frac{2z}{1-z^2}y'(z) + \frac{\nu(\nu+1)}{1-z^2}y(z) = 0 \tag{1}$$

— appear in numerous contexts in physics and applied mathematics. For instance, they arise when certain partial differential equations are solved via separation of variables, they are often used to represent smooth functions defined on bounded intervals, and their roots are the nodes of Gauss–Legendre quadrature rules. For our purposes, it is convenient to work with the functions $\bar{P}_\nu$ and $\bar{Q}_\nu$ defined for $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\nu \geq 0$ via the formulas

$$\bar{P}_\nu(\theta) = \sqrt{\left(\nu + \frac{1}{2}\right)}\, P_\nu(\cos(\theta))\sqrt{\sin(\theta)} \tag{2}$$

and

$$\bar{Q}_\nu(\theta) = -\frac{2}{\pi}\sqrt{\left(\nu + \frac{1}{2}\right)}\, Q_\nu(\cos(\theta))\sqrt{\sin(\theta)}, \tag{3}$$

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http://dx.doi.org/10.1016/j.jcp.2017.08.041

where P_ν and Q_ν are the Legendre functions of the first and second kinds of degree ν, respectively. The functions (2) and (3) are solutions of the second order linear ordinary differential equation

$$y''(\theta) + \left(\left(\nu + \frac{1}{2}\right)^2 + \frac{1}{4}\operatorname{cosec}^2(\theta)\right) y(\theta) = 0 \text{ for all } 0 < \theta < \frac{\pi}{2}. \tag{4}$$

By a slight abuse of terminology, we will refer to (4) as Legendre's differential equation.

The coefficient of y in (4) is positive and increases with ν, with the consequence that $\bar{P}_\nu$ and $\bar{Q}_\nu$ are highly oscillatory when ν is of large magnitude. It has long been known that despite this there exist phase functions for (4) which are nonoscillatory in some sense. In particular, there is a nonoscillatory function α_ν whose derivative is positive on $\left(0, \frac{\pi}{2}\right)$ and such that

$$\bar{P}_\nu(\theta) = \sqrt{W}\, \frac{\cos(\alpha_\nu(\theta))}{\sqrt{\alpha'_\nu(\theta))}} \tag{5}$$

and

$$\bar{Q}_\nu(\theta) = \sqrt{W}\, \frac{\sin(\alpha_\nu(\theta))}{\sqrt{\alpha'_\nu(\theta)}}, \tag{6}$$

where W is the Wronskian

$$W = \frac{2}{\pi}\left(\nu + \frac{1}{2}\right) \tag{7}$$

of the pair $\bar{P}_\nu$, $\bar{Q}_\nu$. By differentiating the expressions

$$\frac{\bar{Q}_\nu(\theta)}{\bar{P}_\nu(\theta)} = \tan\left(\alpha_\nu(\theta)\right) \text{ and } \frac{\bar{P}_\nu(\theta)}{\bar{Q}_\nu(\theta)} = \operatorname{cotan}\left(\alpha_\nu(\theta)\right), \tag{8}$$

at least one of which is sensible at any point in $\left(0, \frac{\pi}{2}\right)$ since $\bar{P}_\nu$ and $\bar{Q}_\nu$ cannot vanish simultaneously there, we obtain

$$\alpha'_\nu(\theta) = \frac{W}{\left(\bar{P}_\nu(\theta)\right)^2 + \left(\bar{Q}_\nu(\theta)\right)^2}. \tag{9}$$

That (9) is nonoscillatory is well known. Indeed, this can be seen in a straightforward fashion from Olver's uniform asymptotic expansions

$$\bar{P}_\nu(\theta) \sim \sqrt{\lambda\theta}\left(J_0(\lambda\theta)\sum_{j=0}^{\infty}\frac{A_j(-\theta^2)}{\lambda^{2j}} - \frac{\theta}{\lambda}J_1(\lambda\theta)\sum_{j=0}^{\infty}\frac{B_j(-\theta^2)}{\lambda^{2j}}\right) \text{ as } \nu\to\infty \tag{10}$$

and

$$\bar{Q}_\nu(\theta) \sim \sqrt{\lambda\theta}\left(Y_0(\lambda\theta)\sum_{j=0}^{\infty}\frac{A_j(-\theta^2)}{\lambda^{2j}} - \frac{\theta}{\lambda}Y_1(\lambda\theta)\sum_{j=0}^{\infty}\frac{B_j(-\theta^2)}{\lambda^{2j}}\right) \text{ as } \nu\to\infty \tag{11}$$

for the Legendre functions (a derivation of these expansions can be found in Chapter 5 of [13]). In (10) and (11), $\lambda = \nu + \frac{1}{2}$, $A_0(\xi) = 1$, and the remaining coefficients $A_1, A_2, \ldots$ and $B_0, B_1, \ldots$ are defined via the formulas

$$B_k(\xi) = -A'_k(\xi) + \frac{1}{|\xi|^2}\int_{\xi}^{0}\left(\frac{1}{16}\left(\operatorname{cosec}^2\left(\sqrt{|\tau|}\right) + \frac{1}{\tau}\right)A_k(\tau) - \frac{A'_k(\tau)}{2\sqrt{|\tau|}}\right)d\tau \tag{12}$$

and

$$A_{k+1}(\xi) = -\xi B'_k(\xi) - \frac{1}{16}\int_{\xi}^{0}\left(\operatorname{cosec}^2\left(\sqrt{|\tau|}\right) + \frac{1}{\tau}\right)B_k(\tau)\,d\tau. \tag{13}$$

By plugging (10) and (11) into (9) and taking (7) into account, we obtain

$$\alpha'_\nu(\theta) = \frac{2}{\pi\theta}\frac{1}{\left(J_0(\lambda\theta)\right)^2 + \left(Y_0(\lambda\theta)\right)^2} + \mathcal{O}\left(\frac{1}{\nu}\right) \text{ as } \nu\to\infty. \tag{14}$$

The function