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Basis adaptation and domain decomposition for steady-state partial differential equations with random coefficients

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ABSTRACT

We present a novel approach for solving steady-state stochastic partial differential equations in high-dimensional random parameter space. The proposed approach combines spatial domain decomposition with basis adaptation for each subdomain. The basis adaptation is used to address the curse of dimensionality by constructing an accurate low-dimensional representation of the stochastic PDE solution (probability density function and/or its leading statistical moments) in each subdomain. Restricting the basis adaptation to a specific subdomain affords finding a locally accurate solution. Then, the solutions from all of the subdomains are stitched together to provide a global solution. We support our construction with numerical experiments for a steady-state diffusion equation with a random spatially dependent coefficient. Our results show that accurate global solutions can be obtained with significantly reduced computational costs.

1. Introduction

Uncertainty quantification for systems with a large number of deterministically unknown input parameters is a formidable computational task. In the probabilistic framework, uncertain parameters are treated as random variables/fields, yielding the governing equations stochastic. The two most popular methods for solving stochastic equations are Monte Carlo (MC) and polynomial chaos (PC) expansion. In both methods, the random input parameters are represented with d random variables using truncated Karhunen–Loève (KL) expansion [1]. MC methods are robust and easy to implement, but they converge at a very slow rate. Hence, they require a large number of samples. On the other hand, the MC convergence rate does not depend on d. Contrary to this, the computational cost of standard PC methods increases exponentially with increasing d, a phenomenon often referred to as the "curse of dimensionality" [2–5]. Because of this, standard PC methods are only efficient for small to moderate d [6–8].

Recently, basis adaptation [9] and active subspace methods [10] have been presented to identify a low-dimensional representation of the solution of stochastic equations. In the current work, we propose a new approach that combines basis adaptation with spatial domain decomposition ([11–15]) to address the problem posed by large d. In particular, we focus on partial differential equations (PDEs) with spatially dependent random coefficients. If considered in the whole spatial domain, the randomness is very high-dimensional. However, a few dominant random parameters could suffice locally [16]. As a result, we can obtain low-dimensional *local* representations of the random solution in each subdomain. To obtain the low dimensional representation in each subdomain, we use the Hilbert space KL expansion [17]. Then, we reconstruct a global solution by stitching together the local solutions for each subdomain.

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The paper is organized as follows: Section 2 presents the problem of uncertainty quantification for PDEs with random coefficients. Section 3 introduces our approach that combines the domain decomposition and basis adaptation. Section 4 contains numerical results for a two-dimensional steady-state diffusion equation for two different types of boundary conditions. Section 5 presents conclusions and ideas for future work.

2. PDEs with random coefficients

Let D be an open subset of $\mathbb{R}^n$ and (Ω, Σ, P) be a complete probability space with sample space Ω, σ-algebra Σ, and probability measure P. We want to find a random field, $u(x,\omega) : D \times \Omega \to \mathbb{R}$, such that P-almost surely

$$\begin{aligned} \mathcal{L}(x, u(x,\omega); a(x,\omega)) &= f(x,\omega) \ \text{ in } D \times \Omega, \\ \mathcal{B}(x, u(x,\omega); a(x,\omega)) &= h(x,\omega) \ \text{ on } \partial D \times \Omega, \end{aligned} \tag{1}$$

where $\mathcal{L}$ is a differential operator and $\mathcal{B}$ is a boundary operator. We model the uncertainty in the stochastic PDE (1) by treating the coefficient $a(x,\omega)$ as a random field and compute the effect of this uncertainty on the solution field $u(x,\omega)$. To solve the stochastic PDE numerically, we discretize the random fields $a(x,\omega)$ and $u(x,\omega)$ in both spatial and stochastic domains. In this paper, we will focus on the special case where $a(x,\omega)$ is a log-normal random field [18]. This means that $a(x,\omega) = \exp[g(x,\omega)]$, where $g(x,\omega)$ is a Gaussian random field whose mean and covariance function are known. We approximate $g(x,\omega)$ with a truncated KL expansion, while the coefficient $a(x,\omega)$ and solution $u(x,\omega)$ are approximated through truncated PC expansions.

2.1. Karhunen–Loève expansion of the random field $g(x,\omega)$

The random field $g(x,\omega)$ can be approximated with a truncated KL expansion [1],

$$g(x,\omega) \approx g(x, \boldsymbol{\xi}(\omega)) = g_0(x) + \sum_{i=1}^{d} \sqrt{\lambda_i}\, g_i(x) \xi_i(\omega), \tag{2}$$

where d is the number of random variables in the truncated expansion; $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_d)^T$, ξ_i are uncorrelated random variables with zero mean; $g_0(x)$ is the mean of the random field $g(x,\omega)$; and $(\lambda_i, g_i(x))$ are eigenvalues and eigenvectors obtained by solving the eigenvalue problem,

$$\int_D C_g(x_1, x_2) g_i(x_2) dx_2 = \lambda_i g_i(x_1), \tag{3}$$

where $C_g(x_1, x_2)$ is the covariance function of the Gaussian random field $g(x,\omega)$. The eigenvalues are positive and non-increasing, and the eigenfunctions $g_i(x)$ are orthonormal,

$$\int_D g_i(x) g_j(x) dx = \delta_{ij}, \tag{4}$$

where δ_{ij} is the Kronecker delta. In this work, we assume the random variables ξ_i have Gaussian distribution. Therefore, ξ_i are independent.

2.2. Polynomial chaos expansion

We approximate the input random field, $a(x,\omega)$ and the solution field $u(x,\omega)$ using truncated PC expansions [19] in Gaussian random variables as follows:

$$a(x,\omega) \approx a(x, \boldsymbol{\xi}(\omega)) = a_0(x) + \sum_{i=1}^{N_\xi} a_i(x) \psi_i(\boldsymbol{\xi}) \tag{5}$$

and

$$u(x,\omega) \approx u(x, \boldsymbol{\xi}(\omega)) \approx u_0(x) + \sum_{i=1}^{N_\xi} u_i(x) \psi_i(\boldsymbol{\xi}), \tag{6}$$

where $N_\xi = \frac{(d+p)!}{d!\ p!} - 1$ is the number of terms in PC expansion for dimension d and order p, $u_0(x)$ is the mean of the solution field, $u_i(x)$ are PC coefficients, and $\{\psi_i(\boldsymbol{\xi})\}$ are multivariate Hermite polynomials. These polynomials are orthogonal with respect to the inner product defined by the expectation in the stochastic space,