Journal of Computational Physics 351 (2017) 216–229

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

An energy-conserving method for stochastic Maxwell equations with multiplicative noise

Jialin Hong [a], Lihai Ji [b], Liying Zhang [c,*], Jiaxiang Cai [d]

[a] Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
[b] Institute of Applied Physics and Computational Mathematics, Beijing 100094, China
[c] School of Mathematical Science, China University of Mining and Technology, Beijing 100083, China
[d] School of Mathematical Science, Huaiyin Normal University, Huai'an, Jiangsu 210046, China

A R T I C L E   I N F O

Article history:
Received 14 July 2016
Received in revised form 20 July 2017
Accepted 18 September 2017
Available online 21 September 2017

Keywords:
Energy-conserving method
Three-dimensional stochastic Maxwell equations
Multiplicative noise
Geometric structure

A B S T R A C T

In this paper, it is shown that three-dimensional stochastic Maxwell equations with multiplicative noise are stochastic Hamiltonian partial differential equations possessing a geometric structure (i.e. stochastic multi-symplectic conservation law), and the energy of system is a conservative quantity almost surely. We propose a stochastic multi-symplectic energy-conserving method for the equations by using the wavelet collocation method in space and stochastic symplectic method in time. Numerical experiments are performed to verify the excellent abilities of the proposed method in providing accurate solution and preserving energy. The mean square convergence result of the method in temporal direction is tested numerically, and numerical comparisons with finite difference method are also investigated.

1. Introduction

In this paper, we consider three-dimensional (3D) stochastic Maxwell equations with multiplicative noise [18]

$$
\begin{aligned}
d\mathbf{E}(\mathbf{t},\mathbf{x},\mathbf{y},\mathbf{z}) &= \nabla\times\mathbf{H}(\mathbf{t},\mathbf{x},\mathbf{y},\mathbf{z})dt-\lambda\mathbf{H}(\mathbf{t},\mathbf{x},\mathbf{y},\mathbf{z})\circ dW(t),\\
d\mathbf{H}(\mathbf{t},\mathbf{x},\mathbf{y},\mathbf{z}) &= -\nabla\times\mathbf{E}(\mathbf{t},\mathbf{x},\mathbf{y},\mathbf{z})dt+\lambda\mathbf{E}(\mathbf{t},\mathbf{x},\mathbf{y},\mathbf{z})\circ dW(t),
\end{aligned}
\tag{1.1}
$$

where $t\in[0,T]$, $(x,y,z)\in\Theta\subset\mathbb{R}^3$, and Θ is a bounded and simply connected domain with smooth boundary $\partial\Theta$. We employ the perfectly electric conducting (PEC) boundary condition

$$
\mathbf{E}\times\mathbf{n}=\mathbf{0}
\tag{1.2}
$$

on $(0,T]\times\partial\Theta$, where $\mathbf{n}$ is the unit outward normal of $\partial\Theta$. The above system is understood in the Stratonovich setting and the symbol $\circ$ stands for the Stratonovich product. Here, $\lambda\geq 0$ measures the size of the noise and W is a Q-Wiener process defined on a given probability space $(\Omega,\mathscr{F},P,\{\mathscr{F}_t\}_{t\in[0,T]})$, with values in the Hilbert space $\mathbb{L}^2(\Theta)$, which is a space of square integrable real-valued functions. Let $\{e_m\}_{m\in\mathbb{N}}$ be an orthonormal basis of $\mathbb{L}^2(\Theta)$ consisting of eigenvectors of a symmetric, nonnegative and finite trace operator Q, i.e., $Tr(Q)=\sum_{m\in\mathbb{N}}\langle Qe_m,e_m\rangle_{L_2}=\sum_{m\in\mathbb{N}}\eta_m<\infty$

* Corresponding author.
E-mail addresses: hjl@lsec.cc.ac.cn (J. Hong), jilihai@lsec.cc.ac.cn (L. Ji), lyzhang@lsec.cc.ac.cn (L. Zhang), thomasjeer@sohu.com (J. Cai).

https://doi.org/10.1016/j.jcp.2017.09.030
0021-9991/

and $Qe_m = \eta_m e_m$. Then there exists a sequence of independent real-valued Brownian motions $\{\beta_m\}_{m\in\mathbb{N}}$ such that $W(t,x,y,z,\omega) = \sum_{m=0}^{\infty} \sqrt{\eta_m}\beta_m(t,\omega)e_m(x,y,z), \quad t \geq 0, (x,y,z) \in \Theta, \omega \in \Omega$.

3D stochastic Maxwell equations with multiplicative noise play an important role in many scientific fields, especially in stochastic electromagnetism and statistical radiophysics [2,8,13,18]. We refer interested readers to [15] for the well-posedness of equations (1.1). By using infinite dimensional Itô formula, it is straightforward to show that the $L^2(\Theta)$-norm of the solution is a constant almost surely (a.s.) (for more details see Theorem 2.2), i.e.,

$$\int_{\Theta} (|\mathbf{E}(x,y,z,t)|^2 + |\mathbf{H}(x,y,z,t)|^2)dxdydz = Constant, \ a.s. \tag{1.3}$$

There have been a lot of ongoing research activities in energy-conserving numerical methods for deterministic Maxwell equations, and various methods have been proposed in the literatures [4,5,14,19]. However, in the stochastic setting, we are only aware of the numerical schemes proposed in [6,9,10,16] for stochastic Hamiltonian ODEs. The authors in [3,11] proposed stochastic multi-symplectic methods for stochastic Maxwell equations with additive noise, which have the merits of preserving the discrete stochastic multi-symplectic conservation law and stochastic energy dissipative properties.

To the best of our knowledge, there has been no reference considering this aspect for stochastic Maxwell equations with multiplicative noise till now. In addition, numerical methods preserving the structure characteristics of equations should be much better in preservation of physical properties and have better stability in numerical computation. Generally speaking, this kind of numerical methods constructed by finite difference techniques is completely implicit for non-separable stochastic system, and demands substantial computational cost. In this paper, we use the idea of wavelet collocation method to get an efficient, multi-symplectic and energy-conserving numerical method. By using this method, we obtain a system of algebraic equations with a sparse differentiation matrix, leading to a numerical algorithm of reduced computational cost. Several numerical examples are presented to show the good behaviors of the proposed method by making comparison with a standard finite difference method. Particularly, numerical experiments demonstrate the mean square convergence order of the proposed numerical method in temporal direction.

The rest of this paper is organized as follows. In section 2, we present some geometric and physical properties of 3D stochastic Maxwell equations with multiplicative noise. It is shown that the phase flow of equations preserves the stochastic multi-symplectic structure of phase space, and the equations possess energy conservation law. In section 3, we propose a numerical method and show that the method preserves discrete energy conservation law and the discrete stochastic multi-symplectic conservation law. In section 4, numerical experiments are performed to testify the effectiveness of the method. Concluding remarks are presented in section 5.

2. Stochastic Maxwell equations

In this section, we will present some preliminary results of 3D stochastic Maxwell equations (1.1). For simplicity in notations, we just consider the case that Wiener process $W(t,x,y,z,\omega)$ is applied in one dimensional spatial direction. Throughout this paper, u_s $(s = t,x,y,z)$ denotes the partial derivative of function u with respect to s, i.e., $u_s = \frac{\partial u}{\partial s}$. $d_t u$ denotes the partial differential of u with respect to t, i.e., $d_t u = \frac{\partial u}{\partial t}dt$.

2.1. Stochastic multi-symplectic structure

A stochastic partial differential equation is called a stochastic Hamiltonian partial differential equation if it can be written in the form [12]

$$Md_t u + K u_x dt = \nabla S_1(u)dt + \nabla S_2(u) \circ dW(t), \ \ u \in \mathbb{R}^d, \tag{2.1}$$

where M and K are skew-symmetric matrices, and S_1 and S_2 are real smooth functions of variable u. Stochastic Maxwell equations with multiplicative noise (1.1) can be written as follows

$$Md_t u + K_1 u_x dt + K_2 u_y dt + K_3 u_z dt = \nabla_u S(u) \circ dW, \ \ u \in \mathbb{R}^6. \tag{2.2}$$

Here,

$$u = \left(H_1, H_2, H_3, E_1, E_2, E_3\right)^T,$$

$$S(u) = \frac{\lambda}{2}\left(|E_1|^2 + |E_2|^2 + |E_3|^2 + |H_1|^2 + |H_2|^2 + |H_3|^2\right)$$

and

$$M = \begin{pmatrix} 0 & -I_{3\times 3} \\ I_{3\times 3} & 0 \end{pmatrix}, \quad K_i = \begin{pmatrix} \mathcal{D}_i & 0 \\ 0 & \mathcal{D}_i \end{pmatrix}, \ \ i = 1,2,3.$$