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Gamblets for opening the complexity-bottleneck of implicit schemes for hyperbolic and parabolic ODEs/PDEs with rough coefficients

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ABSTRACT

Implicit schemes are popular methods for the integration of time dependent PDEs such as hyperbolic and parabolic PDEs. However the necessity to solve corresponding linear systems at each time step constitutes a complexity bottleneck in their application to PDEs with rough coefficients. We present a generalization of gamblets introduced in [62] enabling the resolution of these implicit systems in near-linear complexity and provide rigorous a-priori error bounds on the resulting numerical approximations of hyperbolic and parabolic PDEs. These generalized gamblets induce a multiresolution decomposition of the solution space that is adapted to both the underlying (hyperbolic and parabolic) PDE (and the system of ODEs resulting from space discretization) and to the time-steps of the numerical scheme.

1. Introduction

Implicit schemes are popular and powerful methods for the integration of time dependent PDEs such as hyperbolic and parabolic PDEs [95,44,43,11]. However the necessity to solve corresponding linear systems at each time step constitutes a complexity bottleneck in their application to PDEs with rough coefficients.

Although multigrid methods [34,12,36] have been successfully generalized to time dependent equations [50,96,95,109, 33,105,44], their convergence rate can be severely affected by the lack of regularity of the coefficients [101]. While some degree of robustness can be achieved with algebraic multigrid [80], multilevel finite element splitting [111], hierarchical basis multigrid [6], multilevel preconditioning [97], stabilized hierarchical basis methods [98] and energy minimization [52, 101,108], the design of multigrid/multiresolution methods that are provably robust with respect to rough (L^∞) coefficients was an open problem of practical importance [13] addressed in [62] with the introduction of gamblets (in $\mathcal{O}(N\ln^{3d} N)$ complexity for the first solve and $\mathcal{O}(N\ln^{d+1} N)$ for subsequent solves to achieve grid-size accuracy in H^1-norm for elliptic problems). Numerical evidence suggests the robustness of low rank matrix decomposition based methods such as the Fast Multipole Method [35,110], Hierarchical matrices [37,7] and Hierarchical Interpolative Factorization [42] and while this robustness can be proven rigorously for Hierarchical matrices [7] (the complexity of Hierarchical matrices is $\mathcal{O}(N\ln^{2d+8} N)$ to achieve grid-size accuracy in L^2-norm for elliptic problems [7]) one may wonder if it is possible to rigorously lower this

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known complexity bound and achieve (at the same time) a meaningful multi-resolution decomposition of the solution space for time dependent problems. Although classical wavelet based methods [14,10,28] enable a multi-resolution decomposition of the solution space their performance is also affected by the regularity of coefficients because they are not adapted to the underlying PDEs.

In section 2 we present a generalization of gamblets introduced in [62] and apply them in sections 3 and 4 to the implicit schemes for hyperbolic and parabolic PDEs with rough coefficients. As in [62] these generalized gamblets (1) are elementary solutions of hierarchical information games associated with the process of computing with partial information and limited resources, (2) have a natural Bayesian interpretation under the mixed strategy emerging from the game theoretic formulation, (3) induce a multi-resolution decomposition of the solution space that is adapted to the space-time numerical discretization of the underlying PDE and propagate the solution independently (at each time-step) in each sub-band of the decomposition. The complexity of pre-computing generalized gamblets is $N\ln^{3d} N$ and that of propagating the solution is $N\ln^{d+1} N$ (at each time step, to achieve grid-size accuracy in energy norm). Although real valued gamblets are sufficient for first and second order implicit schemes, higher order implicit schemes may require complex valued gamblets. These complex valued gamblets are introduced and their application to higher order schemes is illustrated in Section 5. Observing that the multiresolution decomposition induced by gamblets has properties that are similar to an eigenspace decomposition, we introduce, in Section 6, a multi-time-step scheme for solving parabolic PDEs (with rough coefficients) in $\mathcal{O}(N\ln^{3d+1} N)$ complexity.

Gamblets are derived from a Game Theoretic approach to Numerical Analysis [62,66] which could be seen as decision theory approach to numerical analysis [100,65]. We refer to the information based complexity literature for an understanding of the natural connection between the notions of computing with partial/priced information and numerical analysis (we refer in particular to [106,74,94,55,107,79,56]). Although statistical approaches to numerical analysis [26,77,88,48,81,47,84,57,58] have, in the past, received little attention, perhaps due to the counterintuitive nature of the process of randomizing a *known* function, the possibilities offered by combining numerical uncertainties/errors with model uncertainties/errors appear to be stimulating their reemergence [19,83,61,41,40,15,23,78,75,66,82]. We refer in particular to [85,83,19] for ODEs and to [61,62,66,20,82] for PDEs. Here the game theoretic approach of [62] is applied to both PDEs and the system of ODEs resulting from their discretization. The multiscale nature of the underlying PDEs results in the stiffness of the corresponding ODEs (these ODEs are not only stiff [91,93,92] they are also characterized by a large range/continuum of time scales [59,60,8]). Although it is natural to integrate such ODEs by an eigenspace decomposition when the dimension of the system of ODEs is small, the cost of such an approach is in general prohibitive. It is to some degree surprising that gamblets have properties that are similar to eigenfunctions, or more precisely Wannier basis functions [103,53] (i.e. linear combinations of eigenfunctions concentrated around a given eigenvalue that are also concentrated in space), while preserving the near-linear complexity of the integration.

Since (see [62]) Gamblets are also natural basis functions for numerical homogenization [104,3,46,30,70,31,9,2,25,90,102,72,51,73,45,76] they can also be employed to achieve sub-linear complexity under sufficient regularity of source terms and initial conditions (see [71,69,72] and Remark 4.3).

We also refer to [66] for a generalization of gamblets to arbitrary continuous linear bijections on Banach spaces (see also [82] for their application to the inversion, compression and approximate PCA of dense kernel matrices at near-linear complexity). As discussed in [66] gamblets also provide a solution to the problem of identifying operator adapted wavelets [21,4,32,17,18,24,1,89,99,87] satisfying three essential properties (see [86,87] for an overview): (a) scale-orthogonality (with respect to the operator scalar product to ensure block-diagonal stiffness matrices), (b) local support (or rapid decay) of the wavelets (to ensures that the individual blocks are sparse) and (c) Riesz stability in the energy norm (to ensure that the blocks are well-conditioned).

2. Gamblets

We will, in this section, present a generalization of the gamblets introduced in [62]. Since the proofs of the results presented in this section are similar to those given in [62] we will refer the reader to [62] and to [66] for these proofs.

2.1. The PDE

Let $\zeta > 0$. Consider the PDE

$$\begin{cases} \frac{4}{\zeta^2}\mu(x)u(x) - \operatorname{div}\big(a(x)\nabla u(x)\big) = g(x) & x \in \Omega; \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ (of arbitrary dimension $d \in \mathbb{N}^*$) with piecewise Lipschitz boundary, a is a symmetric, uniformly elliptic $d \times d$ matrix with entries in $L^\infty(\Omega)$ and such that for all $x \in \Omega$ and $l \in \mathbb{R}^d$,

$$\lambda_{\min}(a)|l|^2 \le l^T a(x) l \le \lambda_{\max}(a)|l|^2, \tag{2.2}$$

and $\mu \in L^\infty(\Omega)$ with for all $x \in \Omega$,