Journal of Computational Physics 348 (2017) 514–533

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A fast numerical scheme for the Godunov–Peshkov–Romenski model of continuum mechanics

Haran Jackson

Cavendish Laboratory, JJ Thomson Ave, Cambridge, CB3 0HE, UK

ARTICLE INFO

Article history:
Received 8 February 2017
Received in revised form 27 June 2017
Accepted 31 July 2017
Available online 3 August 2017

Keywords:
Godunov–Peshkov–Romenski
GPR
Continuum mechanics
Operator splitting
ADER
WENO

ABSTRACT

A new second-order numerical scheme based on an operator splitting is proposed for the Godunov–Peshkov–Romenski model of continuum mechanics. The homogeneous part of the system is solved with a finite volume method based on a WENO reconstruction, and the temporal ODEs are solved using some analytic results presented here. Whilst it is not possible to attain arbitrary-order accuracy with this scheme (as with ADER-WENO schemes used previously), the attainable order of accuracy is often sufficient, and solutions are computationally cheap when compared with other available schemes. The new scheme is compared with an ADER-WENO scheme for various test cases, and a convergence study is undertaken to demonstrate its order of accuracy.

1. Background

1.1. Motivation

The Godunov–Peshkov–Romenski model of continuum mechanics (as described in 1.2) presents an exciting possibility of being able to describe both fluids and solids within the same mathematical framework. This has the potential to streamline development of simulation software by reducing the number of different systems of equations that require solvers, and cutting down on the amount of theoretical work required, for example in the treatment of interfaces in multimaterial problems. In addition to this, the hyperbolic nature of the GPR model ensures that the nonphysical instantaneous transmission of information appearing in certain non-hyperbolic models (such as the Navier–Stokes equations) cannot occur. Parallelization also tends to be easier with hyperbolic models, allowing us to leverage the great advances that have been made in parallel computing architectures in recent years.

At the time of writing, the GPR model has been solved for a variety of fluid and solid problems using the ADER-WENO method (Dumbser et al. [8], Boscheri et al. [4]). ADER-WENO methods (described in 1.3) are extremely effective in producing arbitrarily-high order solutions to hyperbolic systems of PDEs, but in some situations their accompanying computational cost may prove burdensome. A new method is presented in this study that is simple to implement and computationally cheaper than a corresponding ADER-WENO method if only second order accuracy is required. This may prove useful in the design of simulation software addressing problems in which not just accuracy but also speed and usability are of paramount importance.

E-mail address: hj305@cam.ac.uk.

http://dx.doi.org/10.1016/j.jcp.2017.07.055

1.2. The GPR model

The GPR model, first introduced in Peshkov and Romenski [23], has its roots in Godunov and Romenski's 1970s model of elastoplastic deformation (see Godunov and Romenski [14]). It was expanded upon in Dumbser et al. [8] to include thermal conduction. This expanded model takes the following form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_k)}{\partial x_k} = 0 \tag{1a}$$

$$\frac{\partial (\rho v_i)}{\partial t} + \frac{\partial (\rho v_i v_k + p\delta_{ik} - \sigma_{ik})}{\partial x_k} = 0 \tag{1b}$$

$$\frac{\partial A_{ij}}{\partial t} + \frac{\partial (A_{ik} v_k)}{\partial x_j} + v_k \left(\frac{\partial A_{ij}}{\partial x_k} - \frac{\partial A_{ik}}{\partial x_j} \right) = -\frac{\psi_{ij}}{\theta_1(\tau_1)} \tag{1c}$$

$$\frac{\partial (\rho J_i)}{\partial t} + \frac{\partial (\rho J_i v_k + T\delta_{ik})}{\partial x_k} = -\frac{\rho H_i}{\theta_2(\tau_2)} \tag{1d}$$

$$\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho E v_k + (p\delta_{ik} - \sigma_{ik}) v_i + q_k)}{\partial x_k} = 0 \tag{1e}$$

ρ, $\mathbf{v}$, p, δ, σ, T, E, $\boldsymbol{q}$ retain their usual meanings. θ_1 and θ_2 are positive scalar functions, chosen according to the properties of the material being modeled. A is the distortion tensor (containing information about the deformation and rotation of material elements), $\mathbf{J}$ is the thermal impulse vector (a thermal analogue of momentum), τ_1 is the strain dissipation time, and τ_2 is the thermal impulse relaxation time. $\psi = \frac{\partial E}{\partial A}$ and $\boldsymbol{H} = \frac{\partial E}{\partial \boldsymbol{J}}$.

The following definitions are given:

$$p = \rho^2 \frac{\partial E}{\partial \rho} \tag{2a}$$

$$\sigma = -\rho A^T \frac{\partial E}{\partial A} \tag{2b}$$

$$T = \frac{\partial E}{\partial s} \tag{2c}$$

$$\boldsymbol{q} = \frac{\partial E}{\partial s} \frac{\partial E}{\partial \boldsymbol{J}} \tag{2d}$$

To close the system, the equation of state (EOS) must be specified, from which the above quantities and the sources can be derived. E is the sum of the contributions of the energies at the molecular scale (microscale), the material element[1] scale (mesoscale), and the flow scale (macroscale):

$$E = E_1(\rho, p) + E_2(A, \boldsymbol{J}) + E_3(\boldsymbol{v}) \tag{3}$$

The EOS used in this study (and described in the following passages) is taken from Dumbser et al. [8]. It should be noted, however, that this is just one particular choice, and there are many others that may be used.

For an ideal or stiffened gas, E_1 is given by:

$$E_1 = \frac{p + \gamma p_\infty}{(\gamma - 1)\rho} \tag{4}$$

where $p_\infty = 0$ for an ideal gas.

E_2 is chosen to have the following quadratic form:

$$E_2 = \frac{c_s^2}{4} \|\mathrm{dev}(G)\|_F^2 + \frac{\alpha^2}{2} \|\boldsymbol{J}\|^2 \tag{5}$$

c_s is the characteristic velocity of propagation of transverse perturbations. α is a constant related to the characteristic velocity of propagation of heat waves:

$$c_h = \frac{\alpha}{\rho} \sqrt{\frac{T}{c_v}} \tag{6}$$

$G = A^T A$ is the Gramian matrix of the distortion tensor, and $\mathrm{dev}(G)$ is the deviator (trace-free part) of G:

[1] The concept of a *material element* corresponds to that of a fluid parcel from fluid dynamics, applied to both fluids and solids.