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Coupled variational formulations of linear elasticity and the DPG methodology

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A B S T R A C T

This article presents a general approach akin to domain-decomposition methods to solve a single linear PDE, but where each subdomain of a partitioned domain is associated to a distinct variational formulation coming from a mutually well-posed family of *broken* variational formulations of the original PDE. It can be exploited to solve challenging problems in a variety of physical scenarios where stability or a particular mode of convergence is desired in a part of the domain. The linear elasticity equations are solved in this work, but the approach can be applied to other equations as well. The broken variational formulations, which are essentially extensions of more standard formulations, are characterized by the presence of mesh-dependent broken test spaces and interface trial variables at the boundaries of the elements of the mesh. This allows necessary information to be naturally transmitted between adjacent subdomains, resulting in *coupled* variational formulations which are then proved to be globally well-posed. They are solved numerically using the DPG methodology, which is especially crafted to produce stable discretizations of broken formulations. Finally, expected convergence rates are verified in two different and illustrative examples.

1. Introduction

Many equations arising from physical applications can be given a variational formulation, which can then be analyzed using functional analysis, and solved using a discrete version of the formulation and the finite element method [12,29]. In fact, for a fixed set of equations, there may even be multiple variational formulations which are essentially equivalent to the initial equations. Typically, some of these formulations have significant advantages and disadvantages over the other formulations. For example, in linear elasticity, the classical formulation coming from the principle of virtual work is well-known to be computationally efficient to solve with the Galerkin method, but is subject to volumetric locking phenomena for nearly incompressible materials [29]. On the other hand, the Hellinger–Reissner mixed formulation is not as efficient, but it does remain robustly well-posed for nearly incompressible materials and produces a locally conservative stress tensor [22,6].

This article studies the scenario where distinct variational formulations are implemented in different subdomains of the same physical domain. This can be useful in situations where a certain behavior of the equations to be solved is known (or expected) in particular parts of the domain. Hence, in each region one can choose a variational formulation

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which is well-suited to the expected behavior. For example, consider a material with heterogeneous material properties varying within the domain. The properties can vary continuously, as in cloaking applications or biological materials, or discontinuously, as in multi-material problems. Then, in the parts of the domain where it can be an issue (e.g. a nearly incompressible material in linear elasticity), one can choose a variational formulation that is robustly well-posed with respect to the material properties. In the remaining regions, where such robustness is not fundamental, one can choose a more computationally efficient formulation. Another example occurs when a near singularity is expected in a particular area, so that one would hope to use a variational formulation (with possibly an associated adaptivity scheme) which is desirable in that subdomain, but not necessarily in the entire physical domain [34].

The main issue with such an implementation arises at the interfaces between the two subdomains having distinct variational formulations. At this interface, information must pass between the two subdomains to enable communication. This imposes a coupling with both theoretical and practical compatibility issues which can be difficult to resolve and analyze. Moreover, the coupling must be constructed properly so that the entire problem is well-posed. This is not immediate, even if each of the interacting variational formulations is well-posed when considered independently across the whole domain.

At the theoretical and infinite-dimensional level, an attractive possibility that naturally unburdens the compatibility and well-posedness requirements is the use of *broken* variational formulations. These mesh-dependent formulations are extensions of the usual variational formulations to the case involving broken (or discontinuous) test spaces. They first arose in the analysis of the DPG methodology, which is a minimum residual method with broken test spaces [16–18]. As will be seen, a family of distinct broken variational formulations can originate from the same system of equations through variational testing and by integrating by parts in different ways. As expected, the formulations in the family will be closely related to each other. In fact, for many systems of equations, the collection of formulations in each associated family can be shown to be mutually well-posed [9,30,15]. More importantly, the broken formulations in the family will be observed to inescapably possess interface variable unknowns which are a desirable means of communicating the necessary solution variable information across subdomains. This is what allows introducing a proper definition of *coupled* variational formulations, which will later be proved to be globally well-posed.

To actually compute approximate solutions to such a coupled system, one needs a method having discrete trial and test spaces that together retain well-posedness (i.e. numerical stability) [3,5]. This is easily achieved by use of the practical DPG methodology, the very method which motivated the systematic study of broken variational formulations. Indeed, given a discrete trial space, the DPG methodology is especially crafted to approximate an optimal test space which reproduces the stability of the underlying infinite-dimensional broken variational formulation [18]. Due to this unique design, it can be used with often discarded variational formulations which have different trial and test spaces, such as *ultraweak* formulations [18,33,7]. Apart from stability, the methodology carries other significant advantages including a well-behaved a posteriori error estimator for use in adaptive methods and a parallelizable assembly structure allowing local computation of optimal test functions from a standard (yet enriched) discretization of the underlying functional spaces. However, the method is sometimes computationally intensive, because, when compared to standard methods, it typically comes at the cost of adding degrees of freedom along with some extra local computations [32].

The purpose of this work is to demonstrate the use of the DPG methodology in solving the equations of linear elasticity via coupled variational formulations. The family of variational formulations we study was introduced in [30], where all formulations were shown to be simultaneously well-posed. Here, broken formulations will be shown to naturally dispose of many complications arising with the compatibility and well-posedness of coupled variational formulations. Moreover, the use of the DPG methodology will corroborate expected theoretical convergence results. Examples showing the viability of the approach at a practical level will be illustrated, including a case where the demanding scenario of a fully incompressible material is considered. This last case has physical applications in modeling steel braided rubber hoses and even stents.

For the treatment of the DPG methodology as it applies to the equations of linear elasticity, it is worth highlighting [4,26,8,30]. Regarding the coupling of formulations, similarity exists between the approach in this work and that taken in [27] (used for elliptic transmission problems). There, a variational formulation similar to those considered here is coupled with a variational formulation composed of *boundary integral* operators. Afterward, the coupled formulation is discretized with the DPG methodology throughout the entire computational domain. A remark is also warranted for the contributions in [24,25] where the ideas in [27] are extended to couple the DPG methodology with more standard boundary element methods (BEMs), so that different discretization methods are considered across the domain.

This article is organized as follows. In Section 2 a family of variational formulations equivalent to the equations of linear elasticity are introduced, followed by the associated family of broken variational formulations. In Section 3 coupled variational formulations are described. The distinct broken formulations are shown to be compatible across common interfaces and the coupled formulations are proved to be well-posed. In Section 4 the DPG methodology used to solve the coupled formulations is outlined and novel linear algebra improvements are described. Then, in Section 5 two examples are exhibited, numerically solved, and discussed in detail. The last example is a physically-relevant sheathed hose, for which a benchmark exact solution is derived.