Journal of Computational Physics 349 (2017) 1–10

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Comparison of eigenvalue ratios in artificial boundary perturbation and Jacobi preconditioning for solving Poisson equation

Gangjoon Yoon [a], Chohong Min [b,*]

[a] National Institute for Mathematical Sciences, Daejeon 34047, Republic of Korea
[b] Department of Mathematics, Ewha Womans University, Seoul 03760, Republic of Korea

ARTICLE INFO

Article history:
Received 8 December 2016
Received in revised form 2 August 2017
Accepted 5 August 2017
Available online 10 August 2017

Keywords:
Shortley–Weller
Artificial boundary perturbation
Jacobi preconditioner
Condition number
Poisson equation
Finite difference method

ABSTRACT

The Shortley–Weller method is a standard finite difference method for solving the Poisson equation with Dirichlet boundary condition. Unless the domain is rectangular, the method meets an inevitable problem that some of the neighboring nodes may be outside the domain. In this case, an usual treatment is to extrapolate the function values at outside nodes by quadratic polynomial. The extrapolation may become unstable in the sense that some of the extrapolation coefficients increase rapidly when the grid nodes are getting closer to the boundary. A practical remedy, which we call artificial perturbation, is to treat grid nodes very near the boundary as boundary points. The aim of this paper is to reveal the adverse effects of the artificial perturbation on solving the linear system and the convergence of the solution. We show that the matrix is nearly symmetric so that the ratio of its minimum and maximum eigenvalues is an important factor in solving the linear system. Our analysis shows that the artificial perturbation results in a small enhancement of the eigenvalue ratio from $O(1/(h \cdot h_{min})$ to $O(h^{-3})$ and triggers an oscillatory order of convergence. Instead, we suggest using Jacobi or ILU-type preconditioner on the matrix without applying the artificial perturbation. According to our analysis, the preconditioning not only reduces the eigenvalue ratio from $O(1/(h \cdot h_{min})$ to $O(h^{-2})$, but also keeps the sharp second order convergence.

1. Introduction

In this article, we consider the standard finite difference method for solving the Poisson equation $-\Delta u = f$ in a domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary condition $u = g$ on $\partial\Omega$. Let the uniform grid of step size h is denoted by $(h\mathbb{Z})^n$. The discrete domain is then defined as the set of grid nodes inside the domain, i.e. $\Omega^h := \Omega \cap (h\mathbb{Z})^n$.

The standard finite difference method is a dimension-by-dimension application of the central finite difference, and we present mainly the case of one dimension and report any nominal differences in the other dimensions, when required. Unless Ω is rectangular, the method meets an inevitable problem that some of the neighboring nodes may be outside Ω. As

* Corresponding author.
E-mail address: chohong@ewha.ac.kr (C. Min).

http://dx.doi.org/10.1016/j.jcp.2017.08.013
0021-9991/

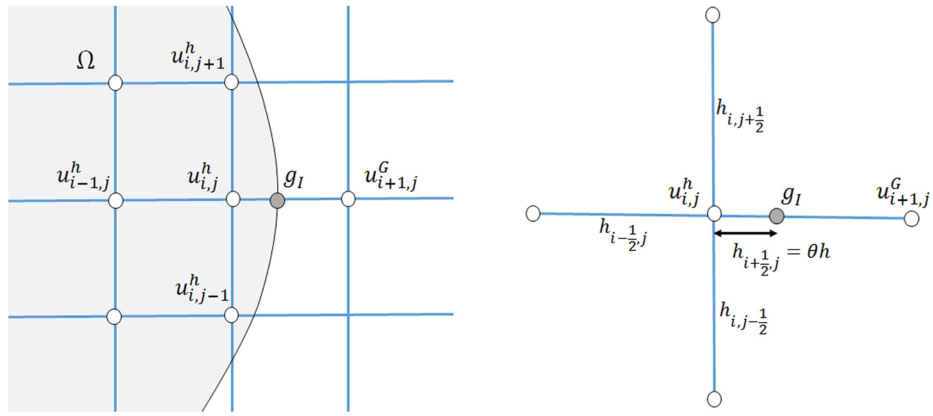

Fig. 1. $u^h_{i,j}$ has four neighboring nodes. The one in the right is outside Ω, and the ghost value $u^G_{i+1,j}$ is quadratically extrapolated from inside values $u^h_{i,j}$ and $u^h_{i-1,j}$ and the boundary value g_I.

depicted in Fig. 1, a neighboring node of the grid node is outside Ω. The node outside Ω^h is called ghost node [10], and the function value at the ghost node is extrapolated by the quadratic polynomial as follows.

$$u^G_{i+1,j} := u^h_{i-1,j}\frac{1-\theta}{1+\theta} - 2u^h_{i,j}\frac{1-\theta}{\theta} + g_I\frac{2}{(1+\theta)\,\theta}.$$

Here $\theta \cdot h$ is the distance between the grid node and the boundary to the right.

Applying the extrapolation to the second-order central difference scheme, we obtain a second-order discretization in the x-direction

$$-(D_{xx}u)_{ij} = -\frac{u_{i-1,j} - 2u_{i,j} + u^G_{i+1,j}}{h^2} = \frac{2}{\theta h \cdot h}u_{i,j} - \frac{2}{h \cdot (\theta+1)h}u_{i-1,j} - \frac{2}{\theta h \cdot (\theta+1)h}g_I. \tag{1}$$

This discretization is called the Shortley and Weller method [19] and the corresponding discrete Laplacian operator is given in equation (3) in section 3. On applying an iterative method to solve the discrete Poisson equation which is related to an nonsymmetric matrix, it is noted in [17] that if the related matrix is nearly symmetric, the residual norm is bounded by the ratio of the maximum and minimum eigenvalues in absolute value. We show in this work that the matrix induced by the Shortley–Weller method is nearly symmetric in the sense that the dominant majority of the entries in $\mathbf{A} = (a_{i,j})$ are symmetric about their diagonals [23] and $a_{ij} \neq 0$ iff $a_{ji} \neq 0$ ([18] and references therein). In this respect, we estimate the convergence performance by the eigenvalue ratio rather than the ratio of singular values.

From the discretization (1), we see that the extrapolation may produce large error if θ in the denominator gets very small. This results in a large condition number for the matrix associated with the Shortley–Weller method as an estimation $|\lambda_{max}/\lambda_{min}| = O\left(1/(h \cdot h_{min})\right)$ shown in Theorem 3.1. Here h_{min} is the minimum distance from the nodes in Ω to the boundary ∂Ω. To mitigate the singularity of the extrapolation, there are two treatments in practice: artificial boundary perturbation and preconditioning.

The artificial boundary perturbation is to treat the grid nodes near the boundary within a certain threshold $\theta_0 \cdot h$ as boundary points and we take $u^h_{i,j} = g_I$ [7,10,16]. The common choice of the threshold is $\theta_0 = h$. We call the practice as the artificial boundary perturbation throughout this paper.

This article is aimed at revealing the precise effects from the artificial perturbation. We review the known facts and estimate the ratio of eigenvalues for the unperturbed linear system in section 2. In section 4, we discuss the effects on the convergence of the numerical solution and the eigenvalue ratio of the linear system for the artificial perturbation. In practice, we take $\theta_0 = h$ for the perturbation value and we reveal in Theorem 4.2 that the eigenvalue ratio to the corresponding treatment is shown to be $O(h^{-3})$ so that the artificial perturbation is less effective than any preconditioning. Another treatment to mitigate the dependence on the minimum grid size and condition numbers as well is preconditioning the linear system. We estimate the effect of the Jacobi preconditioning in section 5 and we test the usual other preconditioners such as symmetric Gauss–Seidel (SGS), incomplete LU (ILU) and modified ILU (MILU) to see the effect of the preconditioning. We show that the Jacobi preconditioning is enough to completely resolve the issue of the singularity of the extrapolation when θ is small, by proving that the Jacobi preconditioner is totally free from the effect of the minimum distance h_{min} and its condition number is no larger than $O(h^{-2})$. Consequently, we suggest the preconditioning method rather that the artificial boundary perturbation in order to improve the performance of the iterative solver.

It is worth noting that it was observed for many second order, self-adjoint, elliptic equations that the spectral condition numbers of the discrete operator grow as $O(h^{-2})$ as the mesh size h tends to zero (see [6] for details). Also, Dupont, Kendall and Rachford [9] observed that even though the convergence rates of the Jacobi, SGS, and ILU preconditioned matrices still