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A high-order relaxation method with projective integration for solving nonlinear systems of hyperbolic conservation laws

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ABSTRACT

We present a general, high-order, fully explicit relaxation scheme which can be applied to any system of nonlinear hyperbolic conservation laws in multiple dimensions. The scheme consists of two steps. In a first (relaxation) step, the nonlinear hyperbolic conservation law is approximated by a kinetic equation with stiff BGK source term. Then, this kinetic equation is integrated in time using a projective integration method. After taking a few small (inner) steps with a simple, explicit method (such as direct forward Euler) to damp out the stiff components of the solution, the time derivative is estimated and used in an (outer) Runge–Kutta method of arbitrary order. We show that, with an appropriate choice of inner step size, the time step restriction on the outer time step is similar to the CFL condition for the hyperbolic conservation law. Moreover, the number of inner time steps is also independent of the stiffness of the BGK source term. We discuss stability and consistency, and illustrate with numerical results (linear advection, Burgers' equation and the shallow water and Euler equations) in one and two spatial dimensions.

1. Introduction

Hyperbolic conservation laws arise in numerous physical applications, such as fluid dynamics, plasma physics, traffic modeling and electromagnetism (see, for instance, [26,37]). They express the conservation of physical quantities (such as mass, momentum, or energy) and may be supplemented with boundary conditions that control influx or outflux at the boundaries of the physical domain [26]. In this paper, we consider a system of hyperbolic conservation laws in multiple spatial dimensions:

$$\partial_t \mathbf{u} + \nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{u}) = \mathbf{0}, \tag{1}$$

or, equivalently,

$$\partial_t \mathbf{u} + \sum_{d=1}^{D} \partial_{x^d} \mathbf{F}^d(\mathbf{u}) = \mathbf{0}, \tag{2}$$

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in which $\mathbf{x} = \left(x^d\right)_{d=1}^{d} \in \mathbb{R}^D$ represents the space variables (D being the number of spatial dimensions), $\mathbf{u}(\mathbf{x}, t) := (u_m(\mathbf{x}, t))_{m=1}^{m} \in \mathbb{R}^M$ denotes the conserved quantities, and $\mathbf{F}(\mathbf{u}) \in \mathbb{R}^{M\times D}$ corresponds to the flux functions.

Hyperbolic conservation laws are often solved using a finite volume method [26,29], which is derived from the integral expression of the conservation law. To that end, in a scalar, one-dimensional setting and with a spatially uniform grid, the domain is divided in I cells $\mathcal{C}_i = [x_{i-1/2}, x_{i+1/2}]$ with constant cell width Δx over which the cell average of the solution $u(x, t)$ to the conservation law

$$\partial_t u + \partial_x F(u) = 0, \tag{3}$$

is approximated at time $t = t^n$ by

$$U_i^n \approx \frac{1}{\Delta x} \int_{\mathcal{C}_i} u(x, t^n) dx. \tag{4}$$

Note that boldface is removed whenever the quantities are scalar. A numerical scheme is then constructed by integrating the conservation law (3) in space over the cell $\mathcal{C}_i$ and in time from t^n to t^{n+1} to obtain

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right), \tag{5}$$

in which $\Delta t = t^{n+1} - t^n$ and the *numerical flux* satisfies

$$F_{i\pm1/2}^n \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} F\left(u(x_{i\pm1/2}, t)\right) dt. \tag{6}$$

Clearly, equation (5) is conservative by construction. The numerical fluxes $F_{i\pm1/2}^n$ can be obtained by constructing an (approximate) Riemann solver, based on a (possibly high-order) reconstruction of the solution in each of the cells using interpolation over the neighboring cells [26,32]. However, in the general nonlinear case, these spatial discretizations require the (possibly tedious) computation of the solutions of local Riemann problems.

Relaxation methods offer an interesting alternative in which the nonlinear hyperbolic conservation law is replaced by a linear transport equation with a stiff nonlinear (but local) source term, see, for instance, discrete kinetic schemes in [18,19,27] and, in particular, [1] which also contains a brief historical overview. In a relaxation method, the conservation law (1) is approximated by a problem of higher dimension containing a small relaxation parameter ε such that, when ε tends to zero, the original problem is recovered. In this paper, we will consider the relaxation problem to be a kinetic BGK equation. In a scalar one-dimensional setting, this equation describes the evolution of a distribution function $f^\varepsilon(x, v, t)$ of particles at position x with velocity v at time t and takes the following form:

$$\partial_t f^\varepsilon + v\partial_x f^\varepsilon = \frac{1}{\varepsilon}\left(\mathcal{M}_v(u^\varepsilon) - f^\varepsilon\right). \tag{7}$$

The left hand side of equation (7) describes the transport of particles, whereas the right hand side represents collisions between particles, which is modeled as a linear relaxation to the Maxwellian $\mathcal{M}_v(u^\varepsilon)$ with a relaxation time ε. The idea is that some of the difficulties associated with the original problem are avoided, while, for sufficiently small ε, the relaxation problem is a good approximation of the problem of interest. In particular, the advantage of the kinetic equation (7) over the conservation law (3) is the fact that the advection term in (7) is now linear, removing the difficulties associated with the high-order discretization of a nonlinear flux term. The disadvantage is the appearance of a stiff source term, which requires special care during time integration. The first methods, proposed in [1,19] are based on splitting techniques. As a consequence, the order in time is restricted to 2 and can only be improved by nontrivial manipulations, see [9]. More recently, several *asymptotic-preserving* methods based on IMEX techniques (in the sense of Jin [17]) have been proposed. An appealing idea along this line of thought, based on IMEX Runge–Kutta methods, is presented in [5,6] for general hyperbolic systems with relaxation. We refer to [10,14] for specific methods for the Boltzmann equation in the hyperbolic and diffusive regimes with a computational cost that is independent of ε. We note that the principle of a kinetic relaxation scheme also bears resemblance to the method of transport [13], see also [38].

In this paper, we propose to use a projective integration method to solve the stiff relaxation systems with an arbitrary order of accuracy in time. We will show that the resulting scheme constitutes a flexible, robust and fully explicit alternative to splitting and IMEX methods, while avoiding the construction of complicated and problem-specific (approximate) Riemann solvers. Projective integration methods were proposed in [15] for stiff systems of ordinary differential equations and analyzed in [23] for kinetic equations with a diffusive scaling. An arbitrary order version, based on Runge–Kutta methods, has been proposed recently in [22], where it was also analyzed for kinetic equations with an advection–diffusion limit. Projective integration is particularly suited for stiff problems with a clear spectral gap. In such stiff problems, the fast modes, corresponding to the Jacobian eigenvalues with large negative real parts, decay quickly, whereas the slow modes correspond to eigenvalues of smaller magnitude and are the solution components of practical interest. Projective integration allows a