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Summation-by-Parts operators with minimal dispersion error for coarse grid flow calculations

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ABSTRACT

We present a procedure for constructing Summation-by-Parts operators with minimal dispersion error both near and far from numerical interfaces. Examples of such operators are constructed and compared with a higher order non-optimised Summation-by-Parts operator. Experiments show that the optimised operators are superior for wave propagation and turbulent flows involving large wavenumbers, long solution times and large ranges of resolution scales.

1. Introduction and motivation

It is well known that maximising the formal order of accuracy with respect to the bandwidth of a finite difference stencil may lead to a suboptimal numerical scheme for problems involving high frequency waves over large time spans. In order to properly resolve such waves, a very small spatial increment is typically required. However, severe restrictions on the spatial increment naturally have a negative impact on the efficiency of the solver. Problems of this type are common in fields such as fluid dynamics, acoustics, meteorology, electromagnetism and seismology, and have been addressed accordingly, see e.g. [1–5]. For these problems, the errors in the numerical solutions are dominated by inaccurate approximations of the dispersion relations related to the governing partial differential equations. Wave properties encoded within the dispersion relation include phase velocity, group velocity, anisotropy and dissipation. It is therefore of interest to develop finite difference schemes that preserve the analytic dispersion relation of the governing equations for a wide range of spatial increments. In [6], a discussion and comparison are made of many stencils that attempt to do so. For more recent approaches, see e.g. [7–9].

Discussions of numerical dispersion have in the literature largely been concerned with central finite difference stencils suitable for purely periodic problems. However, in realistic simulations numerical interfaces will generally be present, at which departure from centrality may be required, and questions of numerical stability must be addressed. In this paper we will present a procedure for designing new finite difference operators on Summation-by-Parts (SBP) form [10–14] that approximate first derivatives both near and far from numerical interfaces. SBP operators combined with weakly imposed interface conditions lead to provably stable schemes, and may be constructed at high orders.

SBP operators based on dispersion relation preserving schemes [15] have previously been constructed in [16] and applied to the Euler and Navier–Stokes equations in [17,18]. However, in these works no particular procedure was applied in order to

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minimise the dispersion near interfaces. In this paper we will construct SBP operators based on the local stencils presented in [9], where the dispersion error of central difference stencils were minimised in the L^∞-sense. We extend this work by analogously minimising the dispersion error of the non-central stencils arising near the interface. This minimisation problem is quasiconvex and may thus be solved fast and reliably using standard optimisation packages.

The SBP operators introduced in this paper are designed for problems where computational resources are a limiting factor. Their intended application is for problems where large wavenumbers are prevalent and grid refinement is not a viable option. As such, it is throughout this paper assumed that some prior knowledge of the spectral content of the solution is available. We will utilise this knowledge in the design of the SBP operators such that performance is improved for problems posed on grids that are coarse relative to the wavenumbers involved. The new operators have suboptimal order of accuracy relative to the bandwidth, leaving a set of free stencil parameters that are used to minimise the dispersion error. Each operator is therefore not intended to be used on a sequence of progressively finer grids. Rather, we introduce a procedure for obtaining a new SBP operator that is adjusted to the particular grid at hand.

This paper is structured as follows: We introduce relevant notation and mathematical concepts in Section 2. In Section 3 we discuss the definition, properties and structure Summation-by-Parts operators. We discuss the procedure of obtaining optimal SBP operators and show a few examples in Section 4. In Section 5 we illustrate the performance of the new SBP operators by solving relevant test problems. Finally, concluding remarks are made in Section 6.

2. Preliminaries

Before we proceed we introduce the notation and theoretical concepts that will be required.

2.1. Notation

Throughout this paper we will separate analytic quantities and approximated quantities, the latter of which we mark with an overbar. Thus, for example $\bar{\xi}$ would represent a numerical approximation of the analytic quantity ξ.

Matrices will in general be denoted by capitalised letters. Functions will be denoted by lower case letters, e.g. $u(x,t)$, unless otherwise specified. If a function is evaluated on a discrete grid we write it as a vector, e.g. $\mathbf{u} = (u_0, \dots, u_N)^T$. Here u_j is the jth element of the $(N+1)$-dimensional vector $\mathbf{u}(t)$, given by the function value of $u(x,t)$ at the point $x = x_j$, $0 \le j \le N$. In this paper we assume that all grids are equidistant and let $\Delta x = x_{j+1} - x_j$ be the spatial increment.

2.2. Dispersion of local finite difference stencils

Consider a local finite difference approximation at the point x_j defined as a weighted sum of function evaluations at L points to the left and R points to the right of x_j;

$$\left(\frac{\partial f}{\partial x}\right)_j \approx \frac{1}{\Delta x}\sum_{m=-L}^{R} c_m f_{j+m}. \tag{1}$$

Here, f_j is the jth element of the vector $\mathbf{f}$, which is obtained by projecting the function $f(x)$ onto the grid, and c_m denote the corresponding weights. Following the derivation in [15], we note that the above equation is a special case of

$$\frac{\partial f(x)}{\partial x} \approx \frac{1}{\Delta x}\sum_{m=-L}^{R} c_m f(x + m\Delta x). \tag{2}$$

when $x = j\Delta x$. By the inverse Fourier transform we may write

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\kappa)\exp(i\kappa x)\mathrm{d}\kappa,$$

where $\hat{f}(\kappa)$ is the Fourier transform of f and κ is a wavenumber. Inserting this into (2), multiplying by $-i\Delta x$ and cancelling common terms gives

$$\Delta x\kappa \approx -i\sum_{m=-L}^{R} c_m \exp(im\Delta x\kappa). \tag{3}$$

Thus, the finite difference stencil provides an approximation of the analytic wavenumber in terms of a truncated Fourier-type series.

We introduce the notation $\xi = \kappa\Delta x$ as the normalised wavenumber. As the smallest resolvable wavelength is $\lambda_{\min} = 2\Delta x$, the largest resolvable wavenumber is $\kappa_{\max} = 2\pi/\lambda_{\min} = \pi/\Delta x$, implying that $|\xi| \le \pi$. Typically we will consider some smaller range of wavenumbers, $\xi \in [0, \xi_{max}]$.