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An entropy stable nodal discontinuous Galerkin method for the two dimensional shallow water equations on unstructured curvilinear meshes with discontinuous bathymetry

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ABSTRACT

We design an arbitrary high-order accurate nodal discontinuous Galerkin spectral element approximation for the non-linear two dimensional shallow water equations with non-constant, possibly discontinuous, bathymetry on unstructured, possibly curved, quadrilateral meshes. The scheme is derived from an equivalent flux differencing formulation of the split form of the equations. We prove that this discretization exactly preserves the local mass and momentum. Furthermore, combined with a special numerical interface flux function, the method exactly preserves the mathematical entropy, which is the total energy for the shallow water equations. By adding a specific form of interface dissipation to the baseline entropy conserving scheme we create a provably entropy stable scheme. That is, the numerical scheme discretely satisfies the second law of thermodynamics. Finally, with a particular discretization of the bathymetry source term we prove that the numerical approximation is well-balanced. We provide numerical examples that verify the theoretical findings and furthermore provide an application of the scheme for a partial break of a curved dam test problem.

1. Introduction

Fluid flows in lakes, rivers, and near coastlines are of interest in oceanography and climate modeling. For such flows the vertical scales of motion are much smaller than the horizontal scales. From this and the assumption of hydrostatic balance [1], the Euler equations can be simplified to the shallow water equations. If the fluid flows over a non-constant bottom topography the shallow water equations may be written as a hyperbolic system of balance laws

$$\vec{w}_t + \vec{f}_x + \vec{g}_y = \vec{s}. \tag{1.1}$$

It is well-known that solutions of the balance laws (1.1) may develop discontinuities in finite time, independent of the smoothness of the initial data. Hence, we consider solutions of the balance laws (1.1) in a weak sense that are well-defined provided the source term $\vec{s}$ remains uniformly bounded, i.e., weak solutions of (1.1) are well-defined under the assumption that the function used to model the bottom topography is in the space $W^{1,\infty}(\mathbb{R})$, see e.g. [2].

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The design of numerical methods to approximate (1.1) is driven by the need for stable, accurate and robust behaviors. For instance, the preservation of steady state solutions is critical in problems with non-constant bottom topographies. Preserving steady solutions discretely is particularly troublesome for discontinuous bottom topographies where special discretization of the source term are required, e.g. [3,4]. One steady state constraint for the shallow water equations is the "lake at rest" condition [4–6], since the relevant waves in a flow can be viewed as small perturbations of the lake at rest, see [5]. A good numerical method for the shallow water equations should accurately capture both steady states and their small perturbations (quasi-steady flows) so as to diminish the appearance of unphysical waves with magnitude proportional to the mesh size (a so-called "numerical storm" [7]), that are normally present for numerical schemes that cannot preserve the "lake at rest" condition. A numerical method that exactly preserves the "lake at rest" steady state property is said to be *well-balanced*, see e.g. [3,5,6,8].

Another critical requirement of the numerics is the ability of the method to remain stable and accurate. Particularly the removal of aliasing errors that can drive non-linear instabilities, and maintenance of stability if discontinuities develop are key characteristics of a numerical scheme. Recent work has appeared on the use of high-order discontinuous Galerkin (DG) approximations to create numerical methods for the solution of systems of conservation laws that discretely satisfy the second law of thermodynamics, e.g., [8–10]. These high-order DG methods may be derived from the perspective of mathematical entropy conservation, e.g. [9,11,12], or reformulations of the PDE into a split formulation to maintain conservation, e.g. [8,10]. We note that the motivation behind these two approaches are similar [13].

The split form of an equation is found by averaging its conservative and non-conservative advective forms. This is problematic as it is not obvious that discretization of the split form remain conservative. However, conservation is critical for the numerical solution to model the correct shock speeds. Recent success has been had using diagonal norm summation-by-parts (SBP) operators to discretize the spatial derivatives in the split formulation of the equations [8,14–17]. Fisher et al. [14] show that split form operators derived from SBP derivative matrices are consistent and conservative in the Lax–Wendroff sense. There is now a known link between SBP finite difference operators and the discontinuous Galerkin spectral element approximation with Legendre–Gauss–Lobatto points, e.g. [15]. This link was used in [8] to derive an entropy conserving discontinuous Galerkin spectral element method (DGSEM) for the one dimensional shallow water equations. This paper exploits the links further and extends in a non-trivial way the previous work found in [8] to multiple dimensions and possibly discontinuous bottom topographies.

In this paper we present an entropy stable, high-order discontinuous Galerkin spectral element approximation for the shallow water equations with a discontinuous bottom topography for unstructured and curved quadrilateral grids. The DGSEM is naturally discontinuous at element boundaries, so we ensure high-order (spectral) accuracy by placing element boundaries at discontinuities in the bottom topography. The ability to do so allows one to model realistic bottom topographies appearing in oceanography. The scheme presented herein is also well-balanced, an attribute difficult to guarantee in curvilinear coordinates. We find that the numerical satisfaction of the metric identities [18] (referred to in [19] as the geometric conservation law) is critical to prove that the baseline scheme remains entropy conservative and well-balanced on arbitrary meshes.

Our approach is to use results of Fisher [19] and Fisher and Carpenter [20] to derive an entropy conserving approximation, and from that an entropy stable one, which is possible because it is possible to reformulate the spectral element approximation of the split form of the shallow water equations into an equivalent flux differencing structure. We use the flux differencing reformulation to prove the underlying properties of the entropy stable DGSEM, as well as to highlight how an existing DGSEM code can be altered to incorporate entropy stability.

The paper is organized as follows: in Sec. 2 we begin with a brief description of the continuous entropy analysis of the two dimensional shallow water equations. We outline the discontinuous Galerkin spectral element method with the summation-by-parts (SBP) property in Sec. 3. This section also introduces the important reformulation of the DGSEM into an equivalent flux differencing framework. We provide in Sec. 4.1 a discretization of the two dimensional shallow water equations using the flux differencing formulation that is conservative and entropy conservative on curvilinear meshes. We also provide a detailed proof that the approximation remains well-balanced. Then, in Sec. 4.2, additional dissipation is added to the scheme to ensure that the approximation remains valid for flow regimes that may contain discontinuities. Numerical results in Sec. 5 demonstrate and underline our theoretical findings. Our conclusions are presented in Sec. 6. Finally, Appendix D provides algorithms and implementation details of how a standard DGSEM code can be altered to incorporate the newly proposed entropy stable fluxes.

2. Shallow water equations

We begin with the balance law form of the two dimensional shallow water equations

$$
\begin{aligned}
h_t + (hu)_x + (hv)_y &= 0,\\
(hu)_t + (h\,u^2 + g\,h^2/2)_x + (huv)_y &= -g\,h\,b_x,\\
(hv)_t + (huv)_x + (h\,v^2 + g\,h^2/2)_y &= -g\,h\,b_y,
\end{aligned}
\tag{2.1}
$$

which includes the continuity and momentum equations. The quantity $h = h(x, y, t)$ denotes the water height measured from the bottom topography $b = b(x, y)$ with the total height given by $H = h + b$. Additionally the constant g is the gravi-