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Arbitrarily high-order time-stepping schemes based on the operator spectrum theory for high-dimensional nonlinear Klein–Gordon equations ☆

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ABSTRACT

In this paper we explore arbitrarily high-order Lagrange collocation-type time-stepping schemes for effectively solving high-dimensional nonlinear Klein–Gordon equations with different boundary conditions. We begin with one-dimensional periodic boundary problems and first formulate an abstract ordinary differential equation (ODE) on a suitable infinity-dimensional function space based on the operator spectrum theory. We then introduce an operator-variation-of-constants formula which is essential for the derivation of our arbitrarily high-order Lagrange collocation-type time-stepping schemes for the nonlinear abstract ODE. The nonlinear stability and convergence are rigorously analysed once the spatial differential operator is approximated by an appropriate positive semi-definite matrix under some suitable smoothness assumptions. With regard to the two dimensional Dirichlet or Neumann boundary problems, our new time-stepping schemes coupled with *discrete Fast Sine / Cosine Transformation* can be applied to simulate the two-dimensional nonlinear Klein–Gordon equations effectively. All essential features of the methodology are present in one-dimensional and two-dimensional cases, although the schemes to be analysed lend themselves with equal to higher-dimensional case. The numerical simulation is implemented and the numerical results clearly demonstrate the advantage and effectiveness of our new schemes in comparison with the existing numerical methods for solving nonlinear Klein–Gordon equations in the literature.

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1. Introduction

In this paper, we begin with the following nonlinear Klein–Gordon equation in a single space variable:

$$\begin{cases} u_{tt} - a^2 \Delta u = f(u), \quad t_0 < t \le T, \ x \in \Omega, \\ u(x, t_0) = \varphi_1(x), \ u_t(x, t_0) = \varphi_2(x), \ x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

and suppose that the initial value problem (1.1) is supplemented with the periodic boundary condition on the domain $\Omega = (-\pi, \pi)$

$$u(x, t) = u(x + 2\pi, t), \tag{1.2}$$

where $u(x, t)$ represents the wave displacement at position x and time t, and $f(u)$ is a nonlinear function of u chosen as the negative derivative of a potential energy $V(u) \ge 0$. In general, there are various choices of the potential function $f(u)$ to investigate solitons and nonlinear phenomena. For instance, the following sine-Gordon equation

$$u_{tt} - a^2 \Delta u + \sin(u) = 0, \tag{1.3}$$

is well known, and other nonlinear potential functions also appear in the literature such as $f(u) = \sinh u$ and polynomial $f(u)$. Moreover, if $u(\cdot, t) \in H^1(\Omega)$ and $u_t(\cdot, t) \in L^2(\Omega)$, the energy conservation becomes another key feature of the Klein–Gordon equation, that is

$$E(t) = \frac{1}{2} \int_{\Omega} \left(u_t^2 + a^2 |\nabla u|^2 + 2V(u) \right) \mathrm{d}x \equiv E(t_0). \tag{1.4}$$

This is an essential property in the theory of solitons. Accordingly, it is also significant to test the effectiveness for a numerical method to preserve the corresponding discrete energy.

In a wide variety of application areas in science and engineering such as nonlinear optics, solid state physics and quantum field theory [10,23,51], the nonlinear wave equation plays an important role and has been investigated by many researchers. In particular, the nonlinear Klein–Gordon equation (1.1) is used to model many different nonlinear phenomena, including the propagation of dislocations in crystals and the behaviour of elementary particles and of Josephson junctions (see Chap. 8.2 in [24] for details). Its description and understanding are of great importance from both the analytical and numerical aspects, and have been investigated by many researchers. Along the analytical front, the Cauchy problem was investigated (see, e.g. [8,13,26,36]). If the energy potential function satisfies $V(u) \ge 0$ for $u \in \mathbb{R}$, the global existence of solutions for the defocussing case was established in [13], and if the energy potential satisfies $V(u) \le 0$ for $u \in \mathbb{R}$, the focusing case possible finite time blow-up was shown in [8]. With regard to the numerical aspect, there have been proposed and studied a variety of solution procedures for solving the nonlinear Klein–Gordon equation. For instance, the energy conservative, explicit, semi-implicit and symplectic conservative standard finite difference time domain (FDTD) discretisations were proposed and analysed in [2,7,25,38,42]. As far as the finite-difference method is concerned, on the basis of standard finite-difference approximations, a three-time level scheme was derived by Strauss and Vázquez in [45]. Jiménez [35] derived conservative finite difference schemes with some analogous discretisations to that used in [45] for the nonlinear term. Other approaches, such as the finite element method and the spectral method, were also studied in [17,19,27,49]. With respect to finite-element techniques, Tourigny [49] proved that the use of product approximations in Galerkin methods subject to Dirichlet boundary conditions dose not affect the convergence rate of the method. Guo et al. [27] developed a conservative Legendre spectral method. Dehghan et al. used the radial basis functions, the dual reciprocity boundary integral equation technique, the collocation and finite difference-collocation methods for solving the nonlinear Klein–Gordon equations or coupled Klein–Gordon equations (see, e.g. [20–22,37]). Although many numerical methods have been derived and investigated for solving the nonlinear Klein–Gordon equation in the literature, in general, the existing numerical methods have limited accuracy, and little attention was paid to the special structure brought by spatial discretisations. This motivates the main theme of this paper to develop arbitrarily high-order Lagrange collocation-type time-stepping schemes for efficiently solving nonlinear Klein–Gordon equations.

The paper is organised as follows. In Section 2, based on the operator spectrum theory, we first formulate the one-dimensional nonlinear Klein–Gordon equation (1.1)–(1.2) as an abstract second order ordinary differential equation on an infinity-dimensional Hilbert space $L^2(\Omega)$. Then, the operator-variation-of-constants formula for the abstract equation is introduced, which is in fact an integral equation of the solution for the nonlinear Klein–Gordon equation (1.1)–(1.2). In Section 3, using the derived operator-variation-of-constants formula, we calculate the nonlinear integrals appeared in this formula by an effective Lagrange interpolation, and then a class of arbitrarily high-order Lagrange collocation-type time-stepping schemes is derived. Furthermore, the investigation of the local error bounds is made, which delivers the simplified order conditions in a much simpler form. Section 4 is devoted to the semidiscretisation. This process enables us to take a subtle but powerful advantage of dealing with the undiscretised differential operator $\mathcal{A}$ and incorporate the special structure introduced by spatial discretisations with the new integrators. The main theoretical results of this work are presented in Section 5. We use the strategy of the energy analysis to study the nonlinear stability and convergence for the fully discrete scheme. Since these fully discrete schemes are implicit, the iterative solutions are required in practical computations.