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# Streamline integration as a method for two-dimensional elliptic grid generation

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ABSTRACT

We propose a new numerical algorithm to construct a structured numerical elliptic grid of a doubly connected domain. Our method is applicable to domains with boundaries defined by two contour lines of a two-dimensional function. Furthermore, we can adapt any analytically given boundary aligned structured grid, which specifically includes polar and Cartesian grids. The resulting coordinate lines are orthogonal to the boundary. Grid points as well as the elements of the Jacobian matrix can be computed efficiently and up to machine precision. In the simplest case we construct conformal grids, yet with the help of weight functions and monitor metrics we can control the distribution of cells across the domain. Our algorithm is parallelizable and easy to implement with elementary numerical methods. We assess the quality of grids by considering both the distribution of cell sizes and the accuracy of the solution to elliptic problems. Among the tested grids these key properties are best fulfilled by the grid constructed with the monitor metric approach.


## 1. Introduction

A structured numerical grid is generated by the (numerical) coordinate transformation of a rectangular “computational domain” to the “physical domain” of interest [1–3]. Compared to unstructured grids a structured grid allows for an easier and computationally cheaper implementation of derivatives since the computational domain is a tensor product grid (a rectangle). Generally, the same numerical techniques as in a Cartesian coordinate system can be used. Still, a structured grid must fulfill certain qualities in order to be useful for practical numerical computations. Among others are smooth and nonoverlapping coordinate lines, boundary orthogonality and a homogeneous distributions of cells across the domain. The latter condition is important in advection schemes. Too small cells deteriorate the CFL condition in explicit schemes and the condition number of the discretization matrix in implicit schemes, while large cells deteriorate the accuracy. Boundary orthogonality is important in order to implement Neumann boundary conditions. As Reference [1] pointed out it is desirable for the grid to be at least near-orthogonal to the boundary. Although it is possible to represent Neumann boundary conditions in a curvilinear grid, the accuracy of the discretization deteriorates and the implementation is more involved than in a boundary-orthogonal grid.

In the literature on grid generation it has been established that elliptic grids are the ones that best fulfill these qualities [1–3]. These grids are generated by a coordinate transformation that fulfills an elliptic equation. However, the generation of these grids based on the algorithm proposed by Thompson, Thames and Mastin (TTM) is numerically involved [3,4].

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A special class of elliptic grids is generated by conformal mappings. The coordinates obey the Cauchy–Riemann differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

They are orthogonal, smooth, and preserve the aspect ratio of the cells from the computational into the physical space. There are many methods to numerically construct a conformal mapping [5], e.g. boundary integral element methods, however their implementation can be tricky. Among further improvements to the original TTM method are grid adaption methods and the monitor metric approach [3,6,7]. Both of these approaches modify the elliptic equation used to construct the coordinate system to a more suitable form. By choosing specific forms the distribution of grid cells is controlled, which makes the coordinate transformation more flexible than conformal mappings. The main numerical difficulty in these techniques is the solution of the nonlinear inverted Beltrami equation. This is required because a (not yet constructed) numerical grid is needed in order to discretize and solve the elliptic equation directly. If the boundary lines are e.g. given by a parametric representation, this conundrum cannot be easily solved. In this contribution we show how this problem can be avoided if the boundary lines of the physical domain are given by a two-dimensional function.

One important application, where this is indeed the case, are tokamaks. These are magnetic fusion devices that use nearly axisymmetric magnetic fields to confine a plasma [8]. In a two-dimensional plane of fixed toroidal angle the magnetic field-lines, in the idealized case, lie within so-called flux-surfaces. These surfaces are in fact the isosurfaces of the poloidal magnetic flux $\psi(x, y)$ described by the Grad–Shafranov equation. A typical numerical modelling scenario involves the study of plasma dynamics between two flux-surfaces $\psi_0$ and $\psi_1$. Structured and unstructured numerical grids have been proposed for the numerical discretization of this region [9–16]. A popular choice are so-called flux or magnetic coordinates [17], in which the magnetic flux $\psi$ or a function of it is the first coordinate. One particular class of such coordinate systems is known as PEST coordinates [12]. In this case, the geometric toroidal angle $\varphi$ is the third coordinate, while a poloidal angle-like coordinate is constructed implicitly by choosing the volume form of the coordinate transformation. Other choices of flux coordinates exist, where the geometric toroidal angle $\varphi$ differs from the toroidal flux angle $\varphi_f$. For example, Boozer [10,11] and Hamada [9] coordinates. In general, a flux aligned coordinate system is not orthogonal however. Flux aligned grids are commonly constructed by integrating the streamlines of the vector field tangential to the isosurfaces of $\psi$ [17,18]. Reference [16] constructed a near-conformal coordinate system that is aligned to the magnetic flux-surfaces in this way. The coordinates are near-conformal in the sense that the grid-deformation is small. Unfortunately, the coordinate lines are not orthogonal.

The previous examples show that it is possible to construct a structured grid of a domain defined by the contour lines of a two-dimensional function $\psi(x, y)$ (the flux). Now recall that this is exactly the requirement for the discretization of an elliptic equation on this domain. The inverted Beltrami equation in the TTM and related methods is no longer needed.

We therefore propose to construct an elliptic grid in three steps. First, a flux aligned grid is constructed by one of the methods that have been proposed in the literature. Then in a second step a suitably chosen elliptic equation is transformed to, discretized and solved in this coordinate system. We investigate the simple Laplace equation, the adapted equation and the elliptic equation with monitor metric suggested by References [3,6,7]. Finally, we treat the solution of the elliptic equation as a new flux function. We can therefore use an adapted version of the algorithm in the first step to construct a second coordinate transformation from the $\psi$-aligned coordinates to coordinates aligned to the solution of the second step. The final transformation then consists of the two consecutive coordinate transformations from the first and the third step.

In this contribution we focus on the discretization of the tokamak edge region as an example. Let us emphasize, however, that the algorithms developed are in no way restricted to magnetic fusion applications. In particular, the monitor metric approach gives sufficient flexibility to handle large classes of problems. For example, in certain pollution models a system of diffusion–advection–reaction equations has to be solved for a given velocity field (the velocity field is determined by analytic modeling, measurements, or computer simulation). In this case the stream function corresponding to the velocity field takes on the role of $\psi$ [19]. Other applications result when a structured, boundary aligned grid is already given analytically, e.g. via Cartesian or polar coordinates. Then the adaption method and the monitor metric approach give very broad control over cell distribution within the domain. This opens a wide range of applications for our algorithms. One application is the improvement of polar coordinates by adaption. This can in particular be useful in simulations of Saturn's north-polar hexagon simulations [20]. Furthermore, Reference [6] shows grids suitable for magnetic reconnection, where the reconnection region has to be highly resolved. Another application could be the simulation of two-dimensional boundary layer flows. The boundary layers typically have to be resolved using a finer mesh than the interior regions, which is for example important in Rayleigh–B'enard convection [21]. The applicability of our algorithm to domains with continuous but not differentiable boundaries (which has interesting applications to resolve coastlines, for example) faces the fundamental problem that the Jacobian does not exist as we approach the boundary. Thus, extreme care has to be taken when applying the streamline integration. We believe that this phenomenon might be related to the problems encountered near X-points (saddle points in $\psi(x, y)$). Of course, if some grid is already available or can be easily constructed (i.e. a Cartesian grid that embeds the domain of interest) we could apply our approach in order to obtain a structured grid. In this case, the monitor metric approach could then be used to adapt this grid for a given problem, which might be more difficult in a Cartesian setting.