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A meshless method for solving two-dimensional variable-order time fractional advection–diffusion equation

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ABSTRACT

Several physical phenomena such as transformation of pollutants, energy, particles and many others can be described by the well-known convection–diffusion equation which is a combination of the diffusion and advection equations. In this paper, this equation is generalized with the concept of variable-order fractional derivatives. The generalized equation is called variable-order time fractional advection–diffusion equation (V-OTFA–DE). An accurate and robust meshless method based on the moving least squares (MLS) approximation and the finite difference scheme is proposed for its numerical solution on two-dimensional (2-D) arbitrary domains. In the time domain, the finite difference technique with a θ-weighted scheme and in the space domain, the MLS approximation are employed to obtain appropriate semi-discrete solutions. Since the newly developed method is a meshless approach, it does not require any background mesh structure to obtain semi-discrete solutions of the problem under consideration, and the numerical solutions are constructed entirely based on a set of scattered nodes. The proposed method is validated in solving three different examples including two benchmark problems and an applied problem of pollutant distribution in the atmosphere. In all such cases, the obtained results show that the proposed method is very accurate and robust. Moreover, a remarkable property so-called positive scheme for the proposed method is observed in solving concentration transport phenomena.

1. Introduction

Fractional or non-integer calculus is concerned with derivatives and integrals of arbitrary orders [1]. It was arisen from a well-known scientific discussion between L'Hopital and Leibniz in 1695, and was subsequently investigated and extended by several famous mathematicians such as Euler, Laplace, Abel, Liouville and Riemann [1]. The subject as a hot issue has drawn the attention of many scientists in mathematics, physics, and engineering in recent years. However, fractional calculus extends the notion of derivative for those cases in which the derivative order is not integer. In recent decades, the use of fractional order derivatives has become extensively attractive in several fields of science and engineering to describe different kinds of problems. The reason for this popularity is that many real-world physical systems display fractional order dynamics and their behavior is governed by fractional differential equations (FDEs) [2]. In other words, FDEs are generalizations of conventional differential equations to an arbitrary order. This kind of equations often appears in different research areas and engineering applications like viscoelasticity, diffusion procedures, relaxation vibrations, electromagnetics,

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electrochemistry etc. [1]. A history of the development of fractional differential operators can be found in [3]. It is worth mentioning that the most important advantage of using FDEs is their nonlocal property [4]. This means that the next state of a dynamical system depends on its current state as well as all of its previous states. Therefore, the memory effect of these derivatives is one of the main justifications for using them in various applications. Thus, FDEs have gained popularity for describing various phenomena such as the ones in [4–7]. Since the order of fractional derivatives and integrals may take any arbitrary value, another extension is considering the order not to be constant. This provides an extension of the classical fractional calculus, namely variable-order fractional calculus. More explicitly speaking, the subject is a generalization of the classical fractional calculus where the orders of fractional derivatives and integrals are functions dependent on the space and/or time. With this approach, the non-local property is more evident and wide applications have been found in many physical models [8–10]. Recently, several researchers such as [11–18] have investigated the same issue and have shown that many complex physical models can be described via variable-order fractional derivatives with a great success. In [11], Sun et al. have investigated the advantages of using variable-order fractional derivatives rather than constant order fractional derivatives. It should be noted that solving variable-order fractional differential equations (V-FDEs) analytically is extremely difficult, and even for most cases impossible due to their high complexity. Thus, in most cases, we need to seek numerical/approximate solutions for them. Hence, proposing efficient and accurate numerical/approximate methods to find numerical solutions for these kinds of problems is highly necessary and important in practice. Some of the recent numerical methods for V-FDEs can be found in [19–30]. Generally speaking, finite-difference methods are nowadays popular for solving various types of V-FDEs. It should be noted that although finite difference methods are very effective for solving various kinds of partial differential equations, the conditional stability of explicit finite difference procedures and the need to use a large amount of CPU time in implicit finite difference schemes limit the applicability of these methods [31]. Furthermore, these methods provide the solution of the problem on mesh points only and accuracy of the techniques is reduced in non-smooth and non-regular domains.

To avoid the mesh generation, meshless (or meshfree) techniques have attracted the attention of researchers in recent years. These methods have become very attractive and efficient for the development of adaptive methods for solving boundary value problems because nodes can be easily added and removed without a burdensome re-meshing of elements [32]. More precisely, a meshless method does not require any mesh or grid to discretize the domain of the problem under consideration, and the approximate solution is constructed entirely based on a set of scattered nodes. It should also be noted that the meshless methods can obtain an accurate and stable solution of integral equations (IEs) or PDEs with various boundary conditions with a set of particles without using any mesh. The most important advantages of meshless methods, as compared with finite element methods, are their high-order continuous shape functions, simpler incorporation of h- and p-adaptivity and certain advantages in crack problems [33]. In recent years, several meshless methods have been considered and developed by researchers to obtain numerical solutions for different types of partial differential equations (PDEs). These methods have been suggested as an alternative numerical tool to eliminate known problems and weaknesses of the traditional mesh-based methods. Some meshless methods are particle methods like SPH [34,35], MPS [36,37], MPE [38], the element-free Galerkin (EFG) method [39], the local radial point interpolation method (LRPIM) [40], the meshless local Petrov–Galerkin (MLPG) method [41], the boundary point interpolation method (BPIM) [42,43], the meshless local boundary integral equation (BIE) method [44], and so on.

The MLS method was first introduced by Shepard [45] to construct smooth approximations for fitting a cloud of points, which was later developed by other researchers. It has successfully been used for surface construction and interpolation of scattered data [46–49]. The method is an effective approach for approximating an unknown function by using a set of disordered data. It consists of a local weighted least square fitting, valid on a small neighborhood of a point and only based on the information provided by its n closest points [50]. It has been found that the MLS method is accurate and stable for arbitrarily distributed nodes for many problems in computational mechanics [51]. Since the numerical approximations of MLS are based on a cluster of scattered nodes instead of interpolation on elements, many meshless methods for the numerical solution of differential equations were based on the MLS method. Having said that, in recent years, some researchers have used a combination of MLS method with other methods such as hybrid finite difference [52], finite elements [53], local meshless method and local radial basis functions [54] and differential quadrature method [55–57]. In recent years, many fractional partial differential equations have been solved using the meshless approach based on the radial basis functions e.g. [58–60]. In [51] the authors presented an implicit meshless approach based on the MLS approximation, using spline weight functions for the numerical simulation of fractional advection–diffusion equation. However, to the best of the authors' knowledge, there is no research to develop the MLS meshless techniques for 2D variable-order fractional PDEs. This issue calls for a significant development.

The main purpose of the present paper is to introduce the variable-order time fractional advection–diffusion equation (V-OTFA–DE) on 2-D arbitrary domains and propose an accurate and robust meshless method based on the MLS approach for its numerical solutions. Therefore, we focus on the following problem:

$$ {}_0^cD_t^{\alpha(\mathbf{x},t)}u(\mathbf{x},t) = \mathcal{K}(\mathbf{x},t)\Delta\, u(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t).\nabla u(\mathbf{x},t) + f(\mathbf{x},t), \quad \mathbf{x}=(x,y)\in\Omega\subset\mathbb{R}^2,\ \ t>0, \tag{1.1} $$

subject to the following general initial and boundary conditions:

$$ \begin{aligned} &u(\mathbf{x},0) = g(\mathbf{x}), && \mathbf{x}\in\Omega,\\ &u(\mathbf{x},t) = h(\mathbf{x},t), && \mathbf{x}\in\partial\Omega,\ \ t>0, \end{aligned} \tag{1.2} $$