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Extending fields in a level set method by solving a biharmonic equation

Timothy J. Moroney *, Dylan R. Lusmore, Scott W. McCue, D.L. Sean McElwain

School of Mathematical Sciences, Queensland University of Technology (QUT), Brisbane QLD 4001, Australia

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ABSTRACT

We present an approach for computing extensions of velocities or other fields in level set methods by solving a biharmonic equation. The approach differs from other commonly used approaches to velocity extension because it deals with the interface fully implicitly through the level set function. No explicit properties of the interface, such as its location or the velocity on the interface, are required in computing the extension. These features lead to a particularly simple implementation using either a sparse direct solver or a matrix-free conjugate gradient solver. Furthermore, we propose a fast Poisson preconditioner that can be used to accelerate the convergence of the latter.

We demonstrate the biharmonic extension on a number of test problems that serve to illustrate its effectiveness at producing smooth and accurate extensions near interfaces. A further feature of the method is the natural way in which it deals with symmetry and periodicity, ensuring through its construction that the extension field also respects these symmetries.

1. Introduction

Moving boundary problems, or interface problems, arise in the mathematical modelling of many interesting natural phenomena, including Hele-Shaw flows [1–4], multiphase fluid flows [5], thin film flows [6–8], melting and solidification problems [9–12], melanoma growth [13] and many others.

Schematics of two particular configurations of a moving boundary problem on a two-dimensional domain D are shown in Fig. 1. We suppose that D is partitioned into two regions, and that the interface $\partial\Omega$ between the two evolves under a normal velocity given by

$$V_n = \mathbf{F} \cdot \mathbf{n} \quad \text{on } \partial\Omega . \tag{1}$$

In the shaded region Ω the velocity field $\mathbf{F}$ is calculated from the solution of a field equation, which we suppose is a parabolic or elliptic partial differential equation. (We do not consider hyperbolic equations in this paper, since as noted by Gibou et al. [14] they require further considerations to ensure that conservation properties are conserved, in order that the correct speed of propagation is calculated in the presence of shocks.) Having calculated the velocity field $\mathbf{F}$, the normal velocity of the interface V_n is then determined from (1). In this way, the interface evolves over time and, depending on the phenomena under consideration, may develop fingering instabilities, splitting, merging, and other interesting features.

* Corresponding author.
E-mail address: t.moroney@qut.edu.au (T.J. Moroney).

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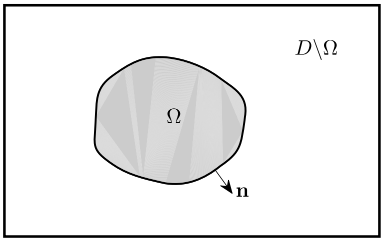
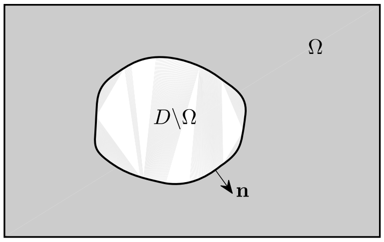


(a) Field known inside, requires extending out



(b) Field known outside, requires extending in

Fig. 1. Schematics of typical moving boundary problems. The interface moves with normal velocity $V_n = \mathbf{F} \cdot \mathbf{n}$. The velocity field $\mathbf{F}$ is calculated in the shaded region Ω from the solution to a field equation. The level set method then requires $\mathbf{F}$ to be smoothly extended to all of D. Here two common scenarios are illustrated, requiring the field to be extended either "out" (a) or "in" (b).

Accurately tracking the evolution of the interface over time is a key requirement of any numerical solution method for such a problem.

The level set method is one numerical method that has proved very popular for solving interface problems [15–17]. Rather than developing an explicit representation for the interface location, the level set method represents the interface *implicitly*, as the zero level set of an auxiliary function ϕ – the so-called level set function. This approach has the distinct advantage that the location of the interface, as well as splitting, merging, and other such complex behaviours as the interface evolves, are all handled naturally by the level set function discretised on a standard rectangular grid. The evolution of ϕ under the external velocity field $\mathbf{F}$ is governed by the level set equation

$$\frac{\partial \phi}{\partial t} + \mathbf{F} \cdot \nabla \phi = 0\,. \tag{2}$$

A typical numerical scheme proceeds by solving the field equation to determine the velocity field, followed by updating the level set according to (2), discretised using finite differences. There is, however, a catch. Although the interfacial velocity V_n has physical significance only on the interface itself, the level set equation (2) requires a field $\mathbf{F}$ defined on all D. Hence, a means of extrapolating, or extending, the velocity field $\mathbf{F}$ from Ω to D is required to complete the level set implementation. This extension velocity need not have any physical significance itself, and is required only to be a smooth, well-defined field on all of D that accurately represents the true velocity in a small neighbourhood of the interface.

Many numerical schemes have been proposed for constructing extension velocities. One approach is to solve the equation

$$\nabla f \cdot \nabla \phi = 0$$

by a process known as the Fast Marching Method [18], which sweeps forward in the upwind direction from the interface, building the extension within an evolving narrow band. This approach is motivated by the desire to maintain the level set ϕ as a signed distance function, satisfying $\|\nabla \phi\| = 1$, under evolution by (2). Higher order variants of the method can also be constructed provided appropriate care is taken with the direction of propagation near the interface for the higher order normal derivatives [19].

An alternative approach to constructing extension velocities is to solve an advection equation

$$\frac{\partial f}{\partial \tau} + H(\phi)\nabla f \cdot \mathbf{n} = 0 \tag{3}$$

to steady-state in pseudo-time τ, where H denotes the Heaviside function [20]. The resulting field is constant along the normal directions, carrying the interfacial value along these characteristics. Higher order schemes can be built upon this foundation by a sequence of extrapolations, first with higher order normal derivatives, through lower order normal derivatives and finally the value itself [21]. Alternatively, the steady version of (3) may be solved directly using a fast sweeping method that carefully chooses and adjusts the required upwind stencils and systematically alternates through different node orderings to cover all characteristic directions [22].

A common feature of these approaches is the need to determine certain explicit properties related to the interface, such as its location, the location of the nearest node to it, or the value of the field on the interface.

This is in contrast with the level set method itself, which deals with the interface in an entirely implicit fashion. Hence we are motivated to pursue a velocity extension approach that also deals with the interface implicitly, and does not rely on an explicit representation of the interface.

A further point we address concerns problems possessing symmetry or periodicity. For these problems, it is natural to exploit this symmetry in the implementation of the level set method by solving over a reduced computational domain. This approach is valid provided the extension field $\mathbf{F}$ *also* possesses the required symmetry. Hence, it is an important consideration when constructing velocity extensions for such problems that they conform to the prescribed symmetry or periodicity conditions on the domain boundary.