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# Reduced Wiener Chaos representation of random fields via basis adaptation and projection

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ABSTRACT

A new characterization of random fields appearing in physical models is presented that is based on their well-known Homogeneous Chaos expansions. We take advantage of the adaptation capabilities of these expansions where the core idea is to rotate the basis of the underlying Gaussian Hilbert space, in order to achieve reduced functional representations that concentrate the induced probability measure in a lower dimensional subspace. For a smooth family of rotations along the domain of interest, the uncorrelated Gaussian inputs are transformed into a Gaussian process, thus introducing a mesoscale that captures intermediate characteristics of the quantity of interest.



## 1. Introduction

Modeling, characterizing and propagating uncertainties in complex physical systems have been extensively explored in recent years as they straddle engineering and the physical, computational, and mathematical sciences. The computational burden associated with a probabilistic representation of these uncertainties is a persistent related challenge. One class of approaches to this challenge has been to seek proper functional representations of the quantities of interest (QoI) under investigation that will be consistent with the observed reality as well as with the mathematical formulation of the underlying physical system, which for instance, is characterized within the context of partial differential equations with stochastic parameters. Additionally, these representations are equipped to serve as accurate propagators useful for prediction or statistical inference purposes. Among the criteria that make such a functional representation a successful candidate, are often the ability to provide a parametric interpretation of the uncertainties involved in a subscale level of the governing physics, as well as its quality as an approximation of what is assumed to be the reality and its discrepancy from it, in terms of several modes of convergence such as distributional, almost sure or functional ($L^2$).

The Homogeneous (Wiener) Chaos [34] representation of random processes has provided a convenient way to characterize solutions of systems of equations that describe physical phenomena as was demonstrated in [13] and further applied to a wide range of engineering problems [9,23,8,10,12]. Generalization of these representations beyond the Gaussian white noise [35,28] provided the foundation for a multi-purpose tool for uncertainty characterization and propagation [19,25,36], statistical updating [27,21,22] and design [15,32] or as a generic mathematical model in order to characterize uncertainties using maximum likelihood techniques [5,14], Bayesian inference [11,2] or maximum entropy [4]. Despite its wide applica-

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bility which has resulted in significant gains, including but not limited to computational efficiencies, its use can still easily become prohibitive with the increase of the dimensionality of the stochastic input. Several attempts using sparse representations [7,6] have only partially managed to sidestep the issue which still remains a major drawback. Recently, a new method for adapted Chaos expansions in Homogeneous Chaos spaces has shown some promising potential as a generic dimensionality reduction technique [30]. The core idea is based on rotating the independent Gaussian inputs through a suitable isometry to form a new basis such that the new expansion expressed in terms of that basis concentrates its probability measure in a lower dimensional subspace, consequently, the basis terms of the Homogeneous Chaos spaces that lie outside that subspace can be filtered out via a projection procedure. Several special cases along with intrusive and non-intrusive computational algorithms were suggested which result in significant model reduction while maintaining high fidelity in the probabilistic characterization of the scalar QoIs.

It is the main objective of the present paper to extend further the basis adaptation technique from simple scalar quantities of interest to random fields or vector valued quantities that admit a polynomial chaos expansion. Such random fields emerge, for instance, as solutions of partial differential equations with random parameters and can be found to have different degree of dependence on the stochastic inputs at different spatio-temporal locations, therefore their adapted representations and the corresponding adapted basis should be expected to exhibit such a spatio-temporal dependence. We provide a general framework where a family of isometries are indexed by the same topological space used for indexing the random field of interest. Several important properties are proved for the new adapted expansion, namely the new stochastic input is no longer a vector of standard normal variables but a Gaussian random field that admits a Karhunen–Loeve [18,20] expansion with respect to those variables. This new quantity essentially merges uncertainties into a new basis that varies at different locations, thus introducing a new way of upscaling uncertainties with localized information about the quantity of interest. In addition, new explicit formulas are derived that allow the transformation of an existing chaos expansion to a new expansion with respect to any chosen basis. One major benefit of this capability is that, once a chaos expansion is available, any suitable adaptation can be achieved without further relying on intrusive and non-intrusive methods that would require additional (repeated) evaluations of the mathematical model, thus delivering us from further computational costs.

This paper is organized as follows: First we introduce the basis adaptation framework for Homogeneous Chaos expansions of random fields that transforms the basis of standard Gaussians into a set of Gaussian random fields. We also recall the two standard ways of choosing the isometry (Gaussian and Quadratic adaptation) that are used for our numerical illustrations. This is followed by the reduction procedure via projection on subspaces of the Hilbert space of square integrable random fields and next we demonstrate how the framework applies when stochasticity is also present in the coefficients of the chaos expansion. At last we provide the theoretical foundations of an infinite dimensional perspective of our approach which shows that our derivations remain consistent and are nothing more but a special case of Hilbert spaces of arbitrary dimension. More specifically our main conclusion of that section is that the components of the newly introduced Gaussian field input are essentially Karhunen–Loeve type expansions of the initial Gaussian variables. Our methodology and theoretical results are illustrated with two numerical examples: That of an elliptic PDE with random diffusion parameter, which explores various ways of obtaining reduced order expansions that adapt well on the random field of interest and an explicit chaos expansion where its first order coefficients consist of a geometric series which allows the comparison of infinite dimensional adaptations and their truncated versions.

## 2. Basis adaptation in Homogeneous Chaos expansions of random fields

### 2.1. The Homogeneous (Wiener) Chaos

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ a $d$-dimensional Gaussian Hilbert space, that is a closed vector space spanned by a set of $d$ independent standard (zero-mean and unit-variance) Gaussian random variables $\{\xi_i\}_{i=1}^d$, equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ defined as $\langle\xi,\zeta\rangle_{\mathcal{G}} = \mathbb{E}[\xi\zeta]$ for $\xi,\zeta\in\mathcal{G}$, where $\mathbb{E}[\cdot]$ denotes the mathematical expectation with respect to the probability measure $\mathbb{P}$. For simplicity, throughout this section we will drop the index $\mathcal{G}$ and simply write $\langle\cdot,\cdot\rangle$ whenever there is no confusion. Let now $\mathcal{F}(\mathcal{G})$ be the $\sigma$-algebra generated by the elements of $\mathcal{G}$, then since all Gaussian variables have finite second moments, it follows that $\mathcal{G}$ is a closed subspace of $L^2(\Omega,\mathcal{F}(\mathcal{G}),\mathbb{P})$. We also define $\mathcal{G}^{\diamond n}$, for $n\in\mathbb{N}\cup\{-1,0\}$ to be the space of all polynomials of exact order $n$, with the convention $\mathcal{G}^{\diamond -1} := \{0\}$. Then clearly $\mathcal{G}^{\diamond 0}$ is the space of constants and $\mathcal{G}^{\diamond 1} = \mathcal{G}$ and in fact from the Cameron–Martin theorem [3,17] we have that $L^2(\Omega,\mathcal{F}(\mathcal{G}),\mathbb{P}) = \bigoplus_{n=0}^{\infty}\mathcal{G}^{\diamond n}$ which has an orthogonal basis that consists of the multidimensional Hermite polynomials defined as

$$\mathbf{h}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \prod_{i=1}^{d} h_{\alpha_i}(\xi_i), \tag{1}$$

where $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_d)\in\mathcal{J} := (\mathbb{N}\cup\{0\})^d$ and $h_{\alpha_i}(\xi_i)$ are the 1-dimensional Hermite polynomials of order $\alpha_i$. More precisely $\{\mathbf{h}_{\boldsymbol{\alpha}}, |\boldsymbol{\alpha}| = n\}$ spans $\mathcal{G}^{\diamond n}$, where $|\boldsymbol{\alpha}| = \sum_i \alpha_i$ and $\boldsymbol{\alpha}! = \prod_{i=1}^{d}\alpha_i!$ and by introducing the orthonormal basis that consists of

$$\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{\mathbf{h}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})}{\sqrt{\boldsymbol{\alpha}!}}, \quad \boldsymbol{\alpha}\in\mathcal{J}, \tag{2}$$