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A subset multicanonical Monte Carlo method for simulating rare failure events

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ABSTRACT

Estimating failure probabilities of engineering systems is an important problem in many engineering fields. In this work we consider such problems where the failure probability is extremely small (e.g. $\leq 10^{-10}$). In this case, standard Monte Carlo methods are not feasible due to the extraordinarily large number of samples required. To address these problems, we propose an algorithm that combines the main ideas of two very powerful failure probability estimation approaches: the subset simulation (SS) and the multicanonical Monte Carlo (MMC) methods. Unlike the standard MMC which samples in the entire domain of the input parameter in each iteration, the proposed subset MMC algorithm adaptively performs MMC simulations in a subset of the state space, which improves the sampling efficiency. With numerical examples we demonstrate that the proposed method is significantly more efficient than both of the SS and the MMC methods. Moreover, like the standard MMC, the proposed algorithm can reconstruct the complete distribution function of the parameter of interest and thus can provide more information than just the failure probabilities of the systems.

1. Introduction

Real-world engineering systems are unavoidably subject to various uncertainties such as material properties, geometric parameters, boundary conditions and applied loadings. These uncertainties may cause undesired events, in particular, system failures or malfunctions, to occur. Accurate evaluation of failure probability of a given system is essential in many engineering fields such as risk management [1], structural safety [2], reliability-based design and optimization [3], and thus is a central task of uncertainty quantification.

Conventionally, the failure probability is often computed by constructing linear or quadratic expansions of the system model around the so-called most probable point or β-point [4], which is known as the first/second order reliability method (FORM/SORM); see e.g., [5] and the references therein. It is well known that FORM/SORM may fail for systems with nonlinearity or multiple failure modes. The Monte Carlo (MC) simulation, which estimates the failure probability by repeatedly simulating the underlying system, provides an alternative to the FORM/SORM methods. The MC method does not make any reduction to the underlying models, and so it can be applied to any systems. On the other hand, it is well known that the MC method suffers from slow convergence, and can become prohibitively expensive when the system failures are rare

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(for example, around 10^{-10}). To this end, many advanced sampling schemes have been developed to reduce the estimation variance and improve the computational efficiency. Among these schemes, the subset simulation (SS) method proposed by Au and Beck [6,7], is one of the most popular sampling strategies for estimating rare failure probabilities. Simply speaking, SS successively constructs a sequence of nested events with the very last one being the event of interest, and the probability of each event is estimated conditionally upon the previous one. Another attractive approach for estimating the failure probability is the multicanonical Monte Carlo (MMC) method [11,12], which was first developed to simulate rare events in physical systems. Later the method was used to estimate rare failure events in optical communication systems [13,14]. More recently, a surrogate accelerated MMC method was developed in [15] for uncertainty quantification applications. The main idea of the MMC method is to partition the state space of the parameter of interest (which is usually a scalar and will be referred to the performance parameter in what follows) into a set of small bins, and then iteratively construct a so-called flat-histogram distribution that can assign equal probabilities into each of the bins. Note that a major advantage of the MMC method is that it can reconstruct the entire distribution function of the parameter of interest, thus can provide with more information than just estimating the probability of a single event. Other small probability estimation methods include, the cross entropy method [8,9], and the population Monte Carlo [10], just to name a few.

In this work, we propose a new algorithm that combines the key ideas of the SS and the MMC methods. Specifically, the new algorithm also constructs a sequence of nested subdomains of the performance parameter, and then performs the MMC scheme in each subdomain. The algorithm preserves some key properties of the standard MMC algorithm, while using the subset idea to accelerate the computation. We thus refer to the proposed algorithm as the subset MMC (SMMC) method in the rest of the work. Like the MMC method, the proposed SMMC algorithm can also compute the entire distribution function of the parameter of interest. Using several examples, we compare the performance of the proposed SMMC algorithm with that of the SS and the MMC methods, and the numerical results show that the new algorithm can significantly outperform both of the original ones.

The rest of the work is organized as the following. In Section 2 we describe the mathematical formulation of the failure probability estimation problem. We then introduce the SS method in Section 3 and the MMC method in Section 4 respectively. The proposed SMMC algorithm is presented in Section 5 and several numerical examples are provided in Section 6. Finally some closing remarks will be given in Section 7.

2. Failure probability estimation

In this section, we shall describe the failure probability estimation problem in a general setting. Consider a probabilistic model where $\mathbf{x}$ is a d-dimensional random variable that represents the uncertainty in the model and the system failure is defined by a real-valued function

$$y = f(\mathbf{x}), \tag{2.1}$$

which is known as the *perform function*. For simplification, we shall assume that the state space of $\mathbf{x}$ is R^d. The event of system failure is defined as that y exceeds a certain threshold value y^*:

$$F = \{\mathbf{x} \in R^d \mid y = f(\mathbf{x}) > y^*\}, \tag{2.2}$$

and as a result the failure probability is

$$P_F = \mathbb{P}(F) = \int_{\{\mathbf{x}\in R^d \mid f(\mathbf{x})>y^*\}} \pi(\mathbf{x})d\mathbf{x} = \int_{\mathbf{x}\in R^d} I_F(\mathbf{x})\pi(\mathbf{x})d\mathbf{x}, \tag{2.3}$$

where $I_A(\mathbf{x})$ is defined as an indicator function of a set A:

$$I_A(\mathbf{x}) = \begin{cases} 1 & \text{if}\, \mathbf{x} \in A, \\ 0 & \text{if}\, \mathbf{x} \notin A; \end{cases}$$

and $\pi(\mathbf{x})$ is the probability density function (PDF) of $\mathbf{x}$. In what follows we shall omit the integration domain when it is simply R^d. This is a general definition for failure probability, which is widely used in many disciplines involving with reliability analysis and risk management. Ideally, P_F can be computed by using the standard MC estimation:

$$P_F \approx \frac{1}{N}\sum_{j=1}^{N} I_F(\mathbf{x}_j), \tag{2.4}$$

where samples $\mathbf{x}_1, ..., \mathbf{x}_N$ are drawn from the distribution $\pi(\mathbf{x})$. However, as has been discussed in Section 1, most engineering systems require high reliability, namely the failure probability $P_F \ll 1$. In this case, MC requires a large number of samples to produce a reliable estimate of P_F. On the other hand, in almost all practical cases, the performance function $f(\mathbf{x})$ does not admit analytical expression and has to be evaluated through expensive computer simulations, which makes the MC estimation of the failure probability prohibitive. Many advanced sampling schemes have been developed to compute the failure probability P_F, and in what follows, we shall introduce two popular choices of them: the SS and the MMC methods.