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Wave dispersion properties of compound finite elements

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ABSTRACT

Mixed finite elements use different approximation spaces for different dependent variables. Certain classes of mixed finite elements, called compatible finite elements, have been shown to exhibit a number of desirable properties for a numerical weather prediction model. In two-dimensions the lowest order element of the Raviart–Thomas based mixed element is the finite element equivalent of the widely used C-grid staggering, which is known to possess good wave dispersion properties, at least for quadrilateral grids. It has recently been proposed that building compound elements from a number of triangular Raviart–Thomas sub-elements, such that both the primal and (implied) dual grid are constructed from the same sub-elements, would allow greater flexibility in the use of different advection schemes along with the ability to build arbitrary polygonal elements. Although the wave dispersion properties of the triangular sub-elements are well understood, those of the compound elements are unknown. It would be useful to know how they compare with the non-compound elements and what properties of the triangular sub-grid elements are inherited?

Here a numerical dispersion analysis is presented for the linear shallow water equations in two dimensions discretised using the lowest order compound Raviart–Thomas finite elements on regular quadrilateral and hexagonal grids. It is found that, in comparison with the well known C-grid scheme, the compound elements exhibit a more isotropic dispersion relation, with a small over estimation of the frequency for short waves compared with the relatively large underestimation for the C-grid. On a quadrilateral grid the compound elements are found to differ from the non-compound Raviart–Thomas quadrilateral elements even for uniform elements, exhibiting the influence of the underlying sub-elements. This is shown to lead to small improvements in the accuracy of the dispersion relation: the compound quadrilateral element is slightly better for gravity waves but slightly worse for inertial waves than the standard lowest order Raviart–Thomas element.

1. Introduction

Traditionally most global atmospheric models used for numerical weather prediction have used a latitude–longitude grid for discretising the equations of motion, though increasingly many modelling groups now use (or are developing) some form of quasi-uniform grid. The latitude–longitude grid has many desirable properties such as orthogonality, symmetry and

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a logically rectangular structure. However, with the increasing number of processor cores expected in future generations of high performance computers, the communication bottleneck implied by the polar singularities in latitude–longitude grids has stimulated the interest in a range of quasi-uniform alternative grids and compact numerical methods. A number of quasi-uniform grids have proved popular in the atmospheric modelling community including: the cubed sphere, (e.g. [22, 27]); subdivision of the icosahedron using triangular (e.g. [14]) and hexagonal elements (e.g. [18,19,9]).

A range of these quasi-uniform alternatives to the latitude–longitude grid for global atmospheric models is reviewed in [21]. They listed a number of essential and desirable properties for an atmospheric model. These can be summarised as requiring the discretisation to: have good conservation properties; mimic certain continuous vector calculus identities; have an accurate representation of balance and adjustment; be free of unphysical modes (either through grid imprinting or computational modes); and have accuracy at least approaching second order.

Cotter and Shipton [5] proposed a number of families of mixed finite elements for quasi-uniform horizontal grids (where mixed refers to the use of different function spaces for the dependent variables, see [2] for a review of mixed elements) which preserve a number of the desirable properties identified by Staniforth and Thuburn [21]. These methods rely upon defining appropriate function spaces $\mathbb{V}_i$ and operator mappings between the spaces. For example, in two dimensions:

$$\mathbb{V}_0 \underset{\overset{\longleftarrow}{\mathbf{k}.\tilde{\nabla}\times}}{\overset{\nabla^{\perp}}{\longrightarrow}} \mathbb{V}_1 \underset{\overset{\longleftarrow}{\tilde{\nabla}}}{\overset{\nabla.}{\longrightarrow}} \mathbb{V}_2, \tag{1}$$

where the $\nabla^{\perp}$ operator is $\boldsymbol{k} \times \nabla$, i.e. the rotation of the gradient operator by 90 degrees anticlockwise with unit vector $\mathbf{k}$ pointing out of the plane. The differential operators along solid lines map from $\mathbb{V}_i \to \mathbb{V}_{i+1}$ e.g. for a vector $\mathbf{w} \in \mathbb{V}_1$, then $\nabla.\mathbf{w} \in \mathbb{V}_2$. The differential operators along dashed lines map from $\mathbb{V}_i \to \mathbb{V}_{i-1}$ in the weak sense obtained via integration by parts, used in (17) and (25) below, for example the weak gradient operator $\tilde{\nabla}$ maps a scalar $\Phi \in \mathbb{V}_2$ to a vector $\tilde{\nabla}\Phi \in \mathbb{V}_1$ and is defined as $\int \boldsymbol{v}.\tilde{\nabla}\Phi da = \oint \boldsymbol{v}.\mathbf{n}\Phi dl - \int (\nabla.\boldsymbol{v})\,\Phi da$ for all $\boldsymbol{v} \in \mathbb{V}_1$. In a shallow water context the streamfunction and potential vorticity $\psi, q \in \mathbb{V}_0$, velocity $\mathbf{u} \in \mathbb{V}_1$ and geopotential $\Phi \in \mathbb{V}_2$. One particular family of finite element complexes suggested by Cotter and Shipton [5] is the family of Raviart–Thomas elements (RT_k) [16] for velocity paired with a continuous bi-polynomial representation of scalars in $\mathbb{V}_0$ $\left(Q_{k+1}\right)$ and a discontinuous bi-polynomial representation of scalars in $\mathbb{V}_2$ $\left(Q_k^{DG}\right)$ denoted $Q_{k+1} - RT_k - Q_k^{DG}$, on quadrilaterals. The lowest order member of this family, $Q_1 - RT_0 - Q_0^{DG}$, corresponds to the mixed finite element analogue of the C-grid finite difference discretisation in that the same number and position of degrees of freedom is obtained. For triangular elements the polynomial space P_k is used instead of the tensor product space Q_k. At the lowest order both P_0^{DG} and Q_0^{DG} represent discontinuous fields that are constant within the element and can be used interchangeably. For notational simplicity the complex of functions spaces $Q_{k+1} - RT_k - Q_k^{DG}$ will be referred by only the vector space RT_k from here on.

At large scales atmospheric motion is dominated by balance and adjustment. Geostrophic and hydrostatic adjustment occur through the emission of inertia-gravity and acoustic waves and the discrete representation of balance can be analysed through the dispersion relation of the candidate numerical scheme. A C-grid staggering, where edge normal velocity components are staggered with respect to the mass variable is commonly used to achieve good dispersion properties, [1]. At lowest order compatible mixed finite elements can be viewed as the finite element generalisation of the C-grid staggering with the flexibility of using the finite element methodology to extend the discretisation to arbitrary order. Although using higher order elements improves the dispersion properties for a range of the spectrum, problems can arise, due to the increased number of branches of solutions, in the form of spectral gaps which can manifested themselves as trapped or distorted waves, for example in the RT_1 [20] and spectral elements [15] methods. In a complete model of the atmosphere the physical parametrisations and boundary conditions can force at scales close the grid scale. Therefore, any unusual behaviour, even if near the limits of resolution, would be of concern. These problems can often be mitigated through various methods such as partial-mass lumping [20], modified quadrature [26] or most commonly diffusion. The dispersion properties of a variety of other mixed elements was discussed by Le Roux [12] (and references therein) to which the interested reader is referred for a more general discussion of mixed finite element dispersion properties. At the lowest order on quadrilaterals, there is a one-to-one mapping of analytical roots to the dispersion relation with the discrete branches (i.e. for the shallow water equations there are two inertia-gravity wave branches and one Rossby wave branch) and therefore spectral gaps are not a problem. However, on non-quadrilateral grids the C-grid staggering leads to a change in the ratio of velocity to mass degrees of freedom, such that there are either too many velocity degrees of freedom (as for a C-grid hexagon) or too many mass degrees of freedom (as for a C-grid triangle). This imbalance gives rise to spurious computational Rossby [23] or inertia-gravity [8] modes respectively. At higher orders the mixed element approach allows the degree of freedom ratio to be chosen so as to retain the desired 2:1 ratio, Cotter and Shipton [5], though this is not a sufficient requirement to obtain good dispersion properties.

A methodology for obtaining mimetic discretisations of the shallow water equations is presented by Cotter and Thuburn [6] using finite element exterior calculus. They propose two methods: termed “primal” and “primal-dual” formulations. The “primal-dual” formulation of Cotter and Thuburn [6] makes use of elements defined on both the primal and dual grid, Fig. 1, in addition to mappings between the corresponding function spaces. As noted in [6] the use of a primal-dual formulation has the advantage over the primal only method of using the dual, discontinuous, representation of potential vorticity, therefore