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High order sub-cell finite volume schemes for solving hyperbolic conservation laws II: Extension to two-dimensional systems on unstructured grids

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ABSTRACT

In this paper, the second in a series, the high order sub-cell finite volume method is extended to two-dimensional hyperbolic systems on unstructured quadrilateral grids. The basic idea of this method is to subdivide a control volume (main cell) into several sub-cells and the finite volume method is applied to each of the sub-cells. The average values on the sub-cells belonging to current and face neighboring main cells are used to reconstruct a common polynomial distribution of a dependent variable on current main cell. This method can achieve high order accuracy using a compact stencil. The focus of this paper is to study the performance of the sub-cell finite volume method on two-dimensional unstructured quadrilateral grids and to verify that high order accuracy can be achieved using face neighboring sub-cells only. Fourier analysis is performed to analyze the dispersion and dissipation properties of the two-dimensional sub-cell finite volume schemes. To capture the discontinuities, the paper proposes a cell-based multi-dimensional limiting procedure using only face-neighboring main cells. Several benchmark test cases are simulated to validate the proposed sub-cell finite volume schemes and the multi-dimensional limiting procedure.

1. Introduction

In this paper, the sub-cell finite volume (SCFV) method proposed in [1] is extended to two-dimensional unstructured grids. The objective of this paper is to develop a numerical method which is robust, high order accurate, and geometrically flexible. The first attempt to develop two-dimensional SCFV schemes has been presented in [2]. In this paper, a detailed analysis of the spectral properties of the SCFV schemes are given and a refined limiting procedure for shock capturing calculation is presented.

Over the last two decades, various unstructured grid high order schemes have been developed, among which the k-exact, ENO and WENO finite volume (FV) methods [3–13], discontinuous Galerkin (DG) methods [14–19], spectral volume (SV) methods [20–23], the multi-moment method [24–31], the $P_N P_M$ [32,33] or hybrid DG/FV methods [34–43] and CPR(FR) method [44–52] are closely related to the current SCFV method.

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Comparing with other high order numerical methods on the unstructured grids, the main advantages of the FV method are relatively simple, computationally efficient and easy to design the implicit time integration schemes. However, the high order FV schemes require a very large stencil in preforming the high order polynomial reconstruction. The large stencil results in a series of problems [34,53] including cache missing, large amount of data need to be transferred at the interface of two partitions in parallel computing, and difficulty in designing high order boundary closure procedures. Recently, Wang et al. [54,55] proposed a class of high order FV schemes on unstructured grid with compact stencils. However, their methods can be efficiently implemented only when being coupled with implicit time integration schemes. An alternative method based on FV discretization with compact stencil is the SV method proposed by Wang et al. [20–23]. SV method can be treated as a discontinuous Petrov–Galerkin method. Instead of using a large stencil of neighboring cells to perform a high-order polynomial reconstruction, the unstructured main cell called the "spectral volume" in SV method is partitioned into a "structured" set of sub-cells, and the average values on these sub-cells are then used to perform a high-order polynomial reconstruction inside the spectral volume. It allows larger CFL number than the same order DG method and is efficient in numerical integration. However, in third, fourth or higher order SV schemes, if the spectral volume is not carefully partitioned, weak instability will occur [56,57]. Additionally, it is rather difficult to design a SV scheme for quadrilateral, hexahedral and triangular prismatic spectral volumes.

The SCFV method can be treated as a combination of the SV method and $P_N P_M$ method. The basic idea of this method is to subdivide a control volume (main cell) into several sub-cells and the finite volume discretization is applied to each of the sub-cells. In the SCFV method, the average values on all the sub-cells of current main cell and all or part of sub-cells of face neighboring main cells are used to reconstruct a common polynomial on current main cell using a weighted least-square method. By a suitable subdivision of sub-cells, the proposed method can achieve high order of accuracy on a compact stencil. However, the use of the least square method in the reconstruction procedure makes the reconstruction polynomial being not conservative for the sub-cell averaged values. A simple correction procedure was then proposed to recover the sub-cell conservation. The Fourier analysis of the one-dimensional SCFV schemes indicated that this correction not only improves the resolution, but also enhances the stability of the SCFV schemes.

In [1], the properties of the proposed high-order SCFV method for solving one dimensional conservation laws were analyzed. This paper extends the basic ideas of SCFV method to two-dimensional unstructured quadrilateral grids and analyzes the Fourier properties of a third order and a fourth order schemes. In Section 2, the framework of the SCFV method is firstly reviewed. In Section 3, a Fourier analysis of the two-dimensional SCFV method is performed on quadrilateral grids. The Fourier analysis section analyzes the dispersion and dissipation properties of the SCFV schemes with different weight parameter in the weighted least square reconstruction and verify the necessity of the correction procedure to recover the sub-cell conservation in two-dimensional cases. Furthermore, we show that the SCFV method can be stable on quadrilateral grids with large aspect ratio or skewness. In Section 4, to capture the discontinuities, the present paper proposes a cell-based multi-dimensional limiting procedure (CMLP). In Section 5, several two-dimensional benchmark cases involving linear and nonlinear equations are used to check the performance of the SCFV method. Finally, conclusions and recommendations for further investigations are summarized in Section 6.

2. Review of the sub-cell finite volume method

In [1], the SCFV scheme was presented for one-dimensional conservation laws. In the present paper, this procedure is reviewed for the case of two-dimensional unstructured grid. We consider the two-dimensional scalar conservation law

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial g(u)}{\partial y}=0 \tag{1}$$

on domain $\Omega \times [0, T]$ with initial condition

$$u(x, y, 0) = u_0(x, y) \tag{2}$$

and appropriate boundary conditions on $\partial\Omega$. In Eq. (1), x and y are the Cartesian coordinates, t denotes time, u is the dependent variable, and $f(u)$ and $g(u)$ are fluxes in x and y directions, respectively. The domain Ω is covered by N non-overlapping control volumes

$$\Omega = \bigcup_{i=1}^{N} \mathrm{CV}_i\,.$$

In SCFV method, each control volume is further subdivided into m sub-cells:

$$\mathrm{CV}_i = \bigcup_{j=1}^{m} \mathrm{SC}_{i;j}\,,$$

where CV_i means the i-th control volume, and $\mathrm{SC}_{i;j}$ means the j-th sub-cell of CV_i. To ease the presentation, the control volume is called the main cell. Here, it should also be noticed that the indexes for the main cell and the sub-cell are separated by semicolon in this paper.