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Fractional spectral and pseudo-spectral methods in unbounded domains: Theory and applications

Hassan Khosravian-Arab [a], Mehdi Dehghan [a,*], M.R. Eslahchi [b]

[a] Department of Applied Mathematics, Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology, No. 424, Hafez Avenue, Tehran, Iran
[b] Department of Applied Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O. Box 14115-134, Tehran, Iran

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ABSTRACT

This paper is intended to provide exponentially accurate Galerkin, Petrov–Galerkin and pseudo-spectral methods for fractional differential equations on a semi-infinite interval. We start our discussion by introducing two new non-classical Lagrange basis functions: NLBFs-1 and NLBFs-2 which are based on the two new families of the associated Laguerre polynomials: GALFs-1 and GALFs-2 obtained recently by the authors in [28]. With respect to the NLBFs-1 and NLBFs-2, two new non-classical interpolants based on the associated- Laguerre–Gauss and Laguerre–Gauss–Radau points are introduced and then fractional (pseudo-spectral) differentiation (and integration) matrices are derived. Convergence and stability of the new interpolants are proved in detail. Several numerical examples are considered to demonstrate the validity and applicability of the basis functions to approximate fractional derivatives (and integrals) of some functions. Moreover, the pseudo-spectral, Galerkin and Petrov–Galerkin methods are successfully applied to solve some physical ordinary differential equations of either fractional orders or integer ones. Some useful comments from the numerical point of view on Galerkin and Petrov–Galerkin methods are listed at the end.

1. Introduction

Roughly speaking, the first study on calculus of integrals and derivatives of an arbitrary order, so called fractional calculus, dated back to the XVII century where Leibniz and L'Hopital were started to study about the meaning of derivative of order 0.5. Subsequent study on fractional calculus was made by other researchers such as Euler in 1730, Lagrange in 1772, Laplace

* Corresponding author.
E-mail addresses: h.khosravian@aut.ac.ir, h.khosravian.arab@gmail.com (H. Khosravian-Arab), mdehghan@aut.ac.ir, mdehghan.aut@gmail.com (M. Dehghan), eslahchi@modares.ac.ir (M.R. Eslahchi).

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in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Grünwald in 1867, Letnikov in 1868, Hadamard in 1892 and Weyl in 1917.

In the past three decades or so, the fractional calculus have found extensive applications in various fields such as physics and engineering, quantum, biophysics, mathematics, biology, gravity and etc.

The topic of fractional calculus also has attracted increasing attention for scientist, engineers and researchers because of their good performance, due to the nonlocal property, to model some interesting problems and phenomena in engineering and science, for instance, diffusion processes [31,32], biophysics [36], viscoelasticity [4,6,12,43], thermodynamics [20], gravity [40], modeling of heat conduction [5], thermal systems [51], biological tissues and control problems [2,22,23,34,45,46]. We also refer the interested reader to [14,33,35,37,38,41,44] for detailed information.

Because of the good performance of fractional (derivatives and integrals) operators to model various physical phenomena, there has been a surge of interest to provide some useful tools to solve problems containing fractional derivatives (or integrals) of Caputo, Riemann–Liouville or Liouville (or Weyl) sense. A comprehensive review of the literatures indicates that there have been a lot of developments in both analytical and numerical approaches to adapt these methods for the problems that have derivatives or integrals of fractional order.

Generally, as far as we know, due to the fact that most problems containing fractional derivatives either don't have analytical (exact) solutions or the exact solutions have very complex forms, so numerical methods for these problems are extensively developed. The numerical methods for such problems can be classified into the local, global and mixed local–global categories. Among the existing numerical approaches, spectral methods – the methods based on orthogonal systems in both finite and infinite intervals – such as tau, Galerkin, Petrov–Galerkin and pseudo-spectral methods are global in character and also have some good features such as: they are usually easy to implement, analyze and perform and oftentimes have high accuracy, *exponentially accuracy*, for the problems with smooth solutions.

To the best knowledge of the authors, until now, researchers have been focused on the use of the classical orthogonal systems such as: trigonometric functions for the periodic problems, Jacobi polynomials and their special cases (Legendre, Chebyshev and Gegenbauer polynomials) for non-periodic problems, associated Laguerre polynomials for problems on the half-line and Hermite polynomials for problems on the whole line. We also note that the classical orthogonal systems are as the eigenfunctions of the usual Sturm–Liouville eigenvalue problems:

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}y(x)\right)+\lambda w(x)y(x)=0,$$

subject to the appropriate boundary conditions. Few years ago, many extensions of the classical orthogonal systems have been considered (see [47] and references therein). But very recently, a new type of fractional Sturm–Liouville eigenvalue problems was introduced by [26,27]. Subsequently, Zayernouri and Karniadakis [54] and the authors of [28] proposed two new classes of orthogonal systems in bounded- and unbounded domains, respectively, which were the eigenfunctions of the fractional Sturm–Liouville eigenvalue problems (see [9,16,21,24,28,29,54] for some detailed information).

It is worthy to note that due to the nature of these classes of orthogonal systems, they can be represented in the form $\{g(x)P_i(x)\}_{i=0}^{\infty}$ where $g(x)$ is given function and $P_i(x)$ are either Jacobi or associated Laguerre polynomials, they have generally non-polynomials natures.

We also point out that with respect to these non-polynomial basis functions, the non-polynomial Lagrange basis functions can be developed in the following form:

$$L_{N,r}(x)=\frac{w(x)}{w(x_r)}h_r(x), \tag{1}$$

where $w(x)$ is some positive weight function and $h_r(x)$ are the Lagrange polynomials associated with the set of collocation nodes $\{x_r\},\ r=0,1,\ldots,N$ which is defined by:

$$h_r(x)=\prod_{\substack{j=0\\ j\neq r}}^{N}\left(\frac{x-x_j}{x_r-x_j}\right),\ r=0,1,\ldots,N. \tag{2}$$

It is easy to check that $L_{N,r}(x)$ satisfies $L_{N,r}(x_k)=\delta_{rk}$ (the Kronecker delta). However, Weideman [52] introduced the non-classical form of the Lagrange basis functions (1) in 1999, but unfortunately, only a few papers dealing with the use of these basis functions. Zayernouri and Karniadakis in [56] considered a non-classical Lagrange basis function (*fractional Lagrange interpolants*) and related fractional pseudo-spectral matrices to solve some fractional ordinary and partial differential equations in bounded domains (see also [55–60]). For the readers' convenience, we summarize the goals of this study as follows:

- The first goal is to introduce two non-classical Lagrange basis functions of the following forms:

$$L^1_{N,r}(x)=\left(\frac{x}{x_r}\right)^{\beta}h_r(x),\ \beta>0,\ L^2_{N,r}(x)=e^{-(x-x_r)}h_r(x),$$