



Diagonal-norm upwind SBP operators



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ABSTRACT

High-order accurate first derivative finite difference operators are derived that naturally introduce artificial dissipation. The boundary closures are based on the diagonal-norm summation-by-parts (SBP) framework and the boundary conditions are imposed using a penalty (SAT) technique, to guarantee linear stability for a large class of initial boundary value problems. These novel first derivative SBP operators have a non-central difference stencil in the interior, and come in pairs (for each order of accuracy). The resulting SBP-SAT approximations lead to fully explicit ODE systems. The accuracy and stability properties are demonstrated for linear first- and second-order hyperbolic problems in 1D, and for the compressible Euler equations in 2D. The newly derived first derivative SBP operators lead to significantly more robust and accurate numerical approximations, compared with the exclusive usage of (previously derived central) non-dissipative first derivative SBP operators.

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1. Introduction

It is well known that higher order methods (as compared to first- and second-order accurate methods) capture wave dominated phenomena more efficiently since they allow a considerable reduction in the degrees of freedom, for a given error tolerance. In particular, high-order finite difference methods (HOFDMs) are ideally suited for problems of this type. (See the pioneering paper by Kreiss and Oliger [13] concerning hyperbolic problems.) The major difficulty with HOFDM is to obtain a stable boundary treatment, something that has received considerable past attention concerning hyperbolic and parabolic problems. (For examples, see [14,33,31,1,3,10].) Roughly speaking, the numerical difficulties increase with the order of the spatial (and temporal) derivatives. For wave-dominated problems, in particular, it is imperative to use finite difference approximations that do not allow growth in time—a property termed “strict stability” [9].

A robust and well-proven high-order finite difference methodology, for well-posed initial boundary value problems (IBVP), is to combine summation-by-parts (SBP) operators [12,32,18] and either the simultaneous approximation term (SAT) method [4], or the projection method [28,29,19,30] to impose boundary conditions (BC). Recent examples of the SBP-SAT approach can be found in [11,16,2,26,8,25].

The SBP operators found in literature (see for example [12,32,18,23,24,5]) are essentially central finite difference stencils closed at the boundaries with a careful choice of one-sided difference stencils, to mimic the underlying integration-by-parts formula in a discrete norm. SBP operators for various derivative orders can be constructed by repeated application of a central-difference first derivative SBP operator, here denoted D_1 . For example, $D_1 D_1$ is a second derivative SBP operator, and is denoted a *wide stencil* second derivative SBP operator in the present study. For linear IBVP with smooth data (here

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referring either to physical data or the underlying curvilinear grid) an SBP-SAT approximation constructed from D_1 operators yields a strictly stable and accurate approximation. For IBVP with non-smooth data (or nonlinear problems), the exclusive usage of D_1 operators in combination with SAT (or projection) does not guarantee strict stability. For nonlinear first order hyperbolic or hyperbolic–parabolic IBVP (such as the Navier–Stokes equation) the addition of artificial dissipation (AD) is most often necessary, to damp spurious oscillations. Adding robust and accurate AD is however far from trivial, and most often involves tuning of “free” parameters. It is imperative that the addition of AD does not destroy the linear stability and accuracy properties or introduce stiffness, which in practice requires a very careful boundary closure of the added AD. How to add AD to traditional (central difference) SBP-SAT approximations based on D_1 was first analysed in [21] and later applied in [37,20], where the notion of odd upwind SBP operators was first introduced.

For second order hyperbolic IBVP a well-known cure to suppress spurious oscillations (when non-smooth features are present) is to employ compatible narrow-stencil second derivative SBP operators (see for example [22,23]), instead of using a wide-stencil approximation based on D_1 . The narrow-stencil second derivative SBP operators are very accurate, and have been successfully implemented in large-scale 2D and 3D problems involving mixed variable coefficient second derivative terms (see for example [39,7]). However, compatible narrow-stencil second derivative SBP operators do not yet exist beyond 6th order accuracy.

The main motivation in [6] for introducing *dual-pair* SBP operators was to introduce damping of spurious oscillations when discretising second order hyperbolic equations. The *dual-pair* SBP operators are first derivative operators with non-central finite difference stencils in the interior. In [6] up to 8th order accurate dual-pair SBP operators are mentioned, but only the 4th order accurate case is presented explicitly. To obtain stability the dual-pair SBP operators are combined with the projection method in [6], to impose BC. In the present study, dual-pair SBP operators with an additional stability constraint are derived, and further combined with the SAT method for imposing BC. These novel SBP operators are here referred to as *upwind* SBP operators since they naturally introduce AD when combined with flux-splitting techniques for hyperbolic systems. (The dual-pair SBP operators in [6] do not have this property.) The novel upwind SBP operators lead to highly robust and accurate discretisations of hyperbolic problems, corroborated through numerical computations of both linear and nonlinear problems. The usefulness for hyperbolic–parabolic systems such as the Navier–Stokes equations, is indicated through stability proofs.

In Section 2 the novel upwind SBP definition is presented, including details of the necessary steps in the derivation. The relationship (and differences) to the previously derived (here referred to as *traditional*) upwind SBP operators in [21] is discussed in Section 3, where the SBP-SAT method is introduced for linear first- and second-order hyperbolic problems in 1D. In Section 4 the accuracy and stability properties of the newly developed upwind SBP operators are verified and compared to previously derived (non-dissipative) central difference SBP operators, by performing numerical simulations. The stability analysis is extended to 2D hyperbolic–parabolic systems in Section 5. Verification of accuracy and stability by numerical long-time simulations of an analytic 2D Euler vortex on a multiblock grid is presented in Section 6. Section 7 summarizes the work. The upwind SBP operators are presented in the Appendix.

2. The finite difference method

Previously derived SBP operators (see for example [12,32,18,23,16,15,24,5]) are essentially central finite difference stencils closed at the boundaries with a careful choice of one-sided difference stencils, to mimic the underlying integration-by-parts formula in a discrete norm. In the present paper the SBP operators are addressed by the accuracy of the interior finite difference stencil. Finite difference SBP operators may be further categorised by the structure of their norm: a) diagonal, b) diagonal interior with block boundary closures, c) fully banded. In the present study the focus is on the derivation of a novel type of diagonal-norm first derivative SBP operator, with a non-central interior stencil. The motivation is to introduce a robust, accurate and parameter free AD with minimal impact on the spectral radius for hyperbolic and hyperbolic–parabolic systems.

2.1. Definitions

The following definitions are needed later in the present study. Let $\mathbf{u}, \mathbf{v} \in L^2[x_l, x_r]$ where $\mathbf{u}^T = [u^{(1)}, u^{(2)}, \dots, u^{(k)}]$ and $\mathbf{v}^T = [v^{(1)}, v^{(2)}, \dots, v^{(k)}]$ are real-valued vector functions with k components. Let the inner product be defined by $(\mathbf{u}, \mathbf{v})_{\mathbf{A}} = \int_{x_l}^{x_r} \mathbf{u}^T \mathbf{A}(x) \mathbf{v} dx$, $\mathbf{A}(x) = \mathbf{A}^T(x) > 0$, and let the corresponding norm be $\|\mathbf{u}\|_{\mathbf{A}}^2 = (\mathbf{u}, \mathbf{u})$.

The domain ($x_l \leq x \leq x_r$) is discretized using the following m equidistant grid points:

$$x_i = x_l + (i - 1)h, \quad i = 1, 2, \dots, m, \quad h = \frac{x_r - x_l}{m - 1}.$$

The following vectors will be used frequently:

$$\begin{aligned} \mathbf{x} &= [x_1, x_2, \dots, x_{m-1}, x_m]^T, & \mathbf{x}^q &= \frac{1}{q!} [x_1^q, x_2^q, \dots, x_{m-1}^q, x_m^q]^T \\ \mathbf{1} &= \mathbf{x}^0 = [1, 1, \dots, 1, 1]^T, & \mathbf{0} &= [0, 0, \dots, 0, 0]^T \\ e_1 &= [1, 0, \dots, 0]^T, & e_m &= [0, \dots, 0, 1]^T, \\ B &= e_m e_m^T - e_1 e_1^T. \end{aligned} \tag{1}$$

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