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Performance analysis of multi-frequency topological derivative for reconstructing perfectly conducting cracks

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Dedicated to Professor Chang Bum Kim on the occasion of his retirement

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ABSTRACT

This paper concerns a fast, one-step iterative technique of imaging extended perfectly conducting cracks with Dirichlet boundary condition. In order to reconstruct the shape of cracks from scattered field data measured at the boundary, we introduce a topological derivative-based electromagnetic imaging functional operated at several nonzero frequencies. The structure of the imaging functionals is carefully analyzed by establishing relationships with infinite series of Bessel functions for the configurations of both symmetric and non-symmetric incident field directions. Identified structure explains why the application of incident fields with symmetric direction operated at multiple frequencies guarantees a successful reconstruction. Various numerical simulations with noise-corrupted data are conducted to assess the performance, effectiveness, robustness, and limitations of the proposed technique.

1. Introduction

One goal of the inverse scattering problem is to find the shape of bulk or flaw in a medium using the scattered field data measured at the boundary. This is an old problem but it has several of applications such as detection of cracks in material engineering, ultrasound imaging in medical sciences, and scanning anti-personnel mines hidden in the ground in military services. Related researches may be found in [1–6] and the references therein. In many studies, various shape reconstruction algorithms have been developed, most of which are focused on the minimization of the least-square functional by using a Newton-type iteration strategy or level-set method [7–10]. The merits of the iteration strategy is that it does not require a very large number of boundary measurements for complete shape reconstruction; instead, in order to guarantee successful reconstruction, it demands very high computational expenditures, optimal regularization terms related to the problems at hand, calculation of a complex Fréchet derivative at each iteration, and *a priori* information of the unknown target to be reconstructed. Nevertheless, even if these conditions are satisfied, the iteration strategy must begin with an initial guess that is close to the true one in order to avoid undesirable situations such as non-convergence or falling into the local minima. Therefore, development of a mathematical theory and an algorithm for generating a good initial guess is an important research topic.

Conversely, certain non-iterative shape reconstruction algorithms have been proposed; for example, MUltiple SIgnal Classification (MUSIC) algorithm [11–18], linear sampling method [19–22], Kirchhoff and subspace migration [11,23–26], and Fourier inversion based one [27–29]. In contrast with the iterative strategy, these algorithms require a large amount of inci-

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dent field data and a significant number of boundary measurements. However, if these conditions are satisfied, non-iterative shape reconstruction algorithms can prove to be effective, fast, and easy to extend to multiple targets.

Topological derivative concept is a non-iterative imaging strategy. Following remarkable works [30–39], this concept was originally developed for the shape optimization problem, but its application to rapid shape reconstruction has only recently been proven, refer to [40–44]. One of the advantages of topological derivative concept is that it does not require a large amount of many incident field data; but, a reduction in the amount of this data has been reported to result in poor resolution of the reconstructed shape. Unfortunately, this fact has been examined via various results of numerical simulations so, development of appropriate mathematical theory must to be considered.

The purpose of this literature is to analyze structure of topological derivative based imaging functional operated at several frequencies for perfectly conducting crack(s) from boundary-measured scattered field data by using only a small amount of incident field data. Based on the derived structure, we discover some intrinsic properties of topological derivative imaging functional, optimal configuration for a successful reconstruction procedure, and smallest number of incident directions for application. It is worth mentioning that for imaging small inclusions, a rigorous justification of the validity of a topological based approach has been considered in [45].

The remainder of this paper is organized as follows. Section 2 briefly introduces the traditional topological derivative by creating a linear, narrow crack. Section 3 describes the design of a multi-frequency topological derivative imaging function and analyzes it to investigate its properties. Section 4 presents various numerical results to assess the performance of the proposed imaging function under various situations. Section 5 contains a short conclusion and some remarks on future work.

2. Topological derivative: an inspection

Assume that a perfectly conducting crack $\mathcal{C}$ is completely hidden in a homogeneous domain $\mathcal{D} \subset \mathbb{R}^2$ with a smooth boundary $\mathcal{B}$. We denote $\vec{\mathbf{x}}_c$ as a two-dimensional vector that lies on crack $\mathcal{C}$.

Throughout this paper, we consider the so-called Transverse Magnetic (TM) polarization case, letting $u^{(m)}(\vec{\mathbf{x}};k)$ be the (single-component) electric field that satisfies the following boundary value problem:

$$\begin{cases} \Delta u^{(m)}(\vec{\mathbf{x}};k) + k^2 u^{(m)}(\vec{\mathbf{x}};k) = 0 & \text{in} \quad \mathcal{D}\backslash\overline{\mathcal{C}} \\ u^{(m)}(\vec{\mathbf{x}};k) = 0 & \text{on} \quad \mathcal{C} \\ \dfrac{\partial u^{(m)}(\vec{\mathbf{x}};k)}{\partial \vec{\boldsymbol{\nu}}(\vec{\mathbf{x}})} = \dfrac{\partial \exp(ik\vec{\boldsymbol{\theta}}_m \cdot \vec{\mathbf{x}})}{\partial \vec{\boldsymbol{\nu}}(\vec{\mathbf{x}})} & \text{on} \quad \mathcal{B}, \end{cases} \tag{1}$$

where $\vec{\boldsymbol{\nu}}(\vec{\mathbf{x}})$ represents the unit outward normal to $\vec{\mathbf{x}} \in \mathcal{B}$ and $\vec{\boldsymbol{\theta}}_m$, $m = 1, 2, \cdots, M$, denotes the two-dimensional vector on the unit circle $\mathbb{S}^1$. Throughout this paper, we let k denote a strictly positive wavenumber and presume k^2 to not be an eigenvalue of (1). Similarly, let $u_{\mathrm{B}}^{(m)}(\vec{\mathbf{x}};k) = \exp(ik\vec{\boldsymbol{\theta}}_m \cdot \vec{\mathbf{x}})$ be the solution of equation (1) without $\mathcal{C}$. Then, the problem we consider here is the computation of the topological derivative of the residual function depending on the solution $u^{(m)}(\vec{\mathbf{x}};k)$:

$$\mathbb{R}(\mathcal{D};k) := \frac{1}{2}\sum_{m=1}^{M} ||u^{(m)}(\vec{\mathbf{x}};k) - u_{\mathrm{B}}^{(m)}(\vec{\mathbf{x}};k)||^2_{L^2(\mathcal{B})} = \frac{1}{2}\sum_{m=1}^{M}\int_{\mathcal{B}} |u^{(m)}(\vec{\mathbf{x}};k) - u_{\mathrm{B}}^{(m)}(\vec{\mathbf{x}};k)|^2 d\mathcal{B}(\vec{\mathbf{x}}), \tag{2}$$

where $|u^{(m)}(\vec{\mathbf{x}};k)|^2 = u^{(m)}(\vec{\mathbf{x}};k)\overline{u^{(m)}(\vec{\mathbf{x}};k)}$.

In order to compute the topological derivative, let us create a small linear crack $\mathcal{L}$ of length $2\ell(\ll k)$ centered at $\vec{\mathbf{x}}_s = (x_1, x_2) \in \mathcal{D}\backslash\mathcal{B}$ such that

$$\mathcal{L} = \{(\xi, x_2) : x_1 - \ell \le \xi \le x_1 + \ell\},$$

and denote $\mathcal{D} \vee \mathcal{L}$ as that domain. Then, because of the change in the topology of $\mathcal{D}$, we can consider the topological derivative $\mathbb{R}_{\mathrm{TD}}(\vec{\mathbf{x}}_s;k)$ based on the residual function $\mathbb{R}(\mathcal{D};k)$ with respect to point $\vec{\mathbf{x}}_s$ as follows:

$$\mathbb{R}_{\mathrm{TD}}(\vec{\mathbf{x}}_s;k) := \lim_{\ell \to 0+} \frac{\mathbb{R}(\mathcal{D} \vee \mathcal{L};k) - \mathbb{R}(\mathcal{D};k)}{\phi(\ell)}, \tag{3}$$

where $\phi(\ell)$ denotes the rate of change of the functional value with respect to the length of the created crack $\mathcal{L}$ and satisfies $\phi(\ell) \longrightarrow 0$ as $\ell \longrightarrow 0+$ (see [34] for instance). From relationship (3), we have the following asymptotic expansion:

$$\mathbb{R}(\mathcal{D} \vee \mathcal{L};k) = \mathbb{R}(\mathcal{D};k) + \phi(\ell)\mathbb{R}_{\mathrm{TD}}(\vec{\mathbf{x}}_s;k) + o(\phi(\ell)).$$

Then, the topological derivative $\mathbb{R}_{\mathrm{TD}}(\vec{\mathbf{x}}_s;k)$ with M different incident waves at a given wavenumber k is as follows (see [46] for its derivation).

Lemma 2.1 *(Topological derivative). Let* $\mathrm{Re}(f)$ *denote the real part of* f*. Then, the function* $\phi(\ell)$ *of (3) and the topological derivative corresponding to (2) are given by*