Journal of Computational Physics 335 (2017) 905–924

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A stochastic asymptotic-preserving scheme for a kinetic-fluid model for disperse two-phase flows with uncertainty

Shi Jin [a,b,*,1], Ruiwen Shu [a]

[a] Department of Mathematics, University of Wisconsin–Madison, Madison, WI 53706, USA
[b] Institute of Natural Sciences, School of Mathematical Science, MOELSEC and SHL-MAC, Shanghai Jiao Tong University, Shanghai 200240, China

ARTICLE INFO

Article history:
Received 2 October 2016
Received in revised form 24 January 2017
Accepted 25 January 2017
Available online 1 February 2017

Keywords:
Kinetic-fluid model
Disperse particles
Uncertain quantification
Asymptotic-preserving
Projection method
Stochastic Galerkin

ABSTRACT

In this paper we consider a kinetic-fluid model for disperse two-phase flows with uncertainty. We propose a stochastic asymptotic-preserving (s-AP) scheme in the generalized polynomial chaos stochastic Galerkin (gPC-sG) framework, which allows the efficient computation of the problem in both kinetic and hydrodynamic regimes. The s-AP property is proved by deriving the equilibrium of the gPC version of the Fokker–Planck operator. The coefficient matrices that arise in a Helmholtz equation and a Poisson equation, essential ingredients of the algorithms, are proved to be positive definite under reasonable and mild assumptions. The computation of the gPC version of a translation operator that arises in the inversion of the Fokker–Planck operator is accelerated by a spectrally accurate splitting method. Numerical examples illustrate the s-AP property and the efficiency of the gPC-sG method in various asymptotic regimes.

1. Introduction

In this paper, we are concerned with kinetic-fluid models for disperse two-phase flows. Such models arise naturally in the study of mixture of a continuum of fluid, such as gas and liquids, and small particles, such as droplets and suspension of solids. The fluid phase is described by hydrodynamic equations, such as the Euler equations or the Navier–Stokes equations, while the particle phase is described by a kinetic equation. The application of kinetic-fluid models includes the dynamic of sprays [7,25,17], granular flows [1,9,6], and combustion theory [10,26], to name a few.

We focus on the model where the fluid phase has small compressibility and nonzero viscosity, thus modeled by the incompressible Navier–Stokes (NS) equations. The particles are assumed to be subjected to a drag force obeying the Stokes Law, i.e., proportional to the relative velocity of the particle and the fluid. Furthermore, the particles are subject to an external force field (gravity, for example) and Brownian motions. For simplicity we take the physical space to be 2-dimensional. The model is given by

* Corresponding author.
E-mail addresses: sjin@wisc.edu (S. Jin), rshu2@math.wisc.edu (R. Shu).

[1] This author's research was supported by NSF grants DMS-1522184 and DMS-1107291: RNMS KI-Net, by NSFC grant No. 91330203, and by the Office of the Vice Chancellor for Research and Graduate Education at the University of Wisconsin–Madison with funding from the Wisconsin Alumni Research Foundation.

http://dx.doi.org/10.1016/j.jcp.2017.01.059
0021-9991/

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\Phi\cdot\nabla_v f = \dfrac{1}{\epsilon}\mathcal{L}_u f, \\ \partial_t u + \nabla_x\cdot(u\otimes u) + \nabla_x p - \dfrac{1}{Re}\Delta_x u = \dfrac{1}{\epsilon}\kappa\displaystyle\int (v-u) f\,\mathrm{d}v, \\ \nabla_x\cdot u = 0, \end{cases} \tag{1.1}$$

where $\mathcal{L}_u f$ is the Fokker–Planck (FP) operator

$$\mathcal{L}_u f = \nabla_v\cdot((v-u)f + \nabla_v f). \tag{1.2}$$

$x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ is the space variable, and $v = (v_1, v_2) \in \mathbb{R}^2$ is the velocity variable. $f = f(t, x, v)$ is the density function of the particles. $u = u(t, x) = (u^{(1)}(t, x), u^{(2)}(t, x))$ is the velocity field of the fluid. $\Phi = \Phi(x)$ is an external force field. The first equation describes the motion of particles. The two terms in the FP operator comes from the drag force from the fluid and the effect of Brownian motions, respectively. The $\nabla_x\Phi$ term is the effect of the external force field on the particles. The second and third equations are the standard Navier–Stokes equations for the fluid, with the right-hand-side term describing the force coming from the particle. $\kappa > 0$ is the coupling constant depending on the strength of interaction between the particles and the fluid, and Re is the Reynolds number. ϵ is the Knudsen number given by $\epsilon = \frac{2\rho_P a^2}{9\mu}$, where μ is the dynamic viscosity of the fluid, a the typical radius of the particles, and ρ_P the density of the particles.

The hydrodynamic limit of this model was first investigated by Goudon et al. [11,12] in two different regimes. We follow the scaling given in [12], which is called the fine particle regime.

In [13] Goudon et al. proposed an Asymptotic-Preserving (AP) [18] scheme for the two-phase flow system (1.1), which are efficient for both the cases of small and large ϵ. The main idea of this work is to incorporate the evolution of the moments of the particles into the projection method [5,27] for the NS system. The possibly stiff (when ϵ is small) FP operator is treated fully implicitly, with a well-balanced spatial discretization proposed by Jin and Yan [19]. The second order time discretization is given by the backward difference.

The paper [13] only concerns with the case where all the physical quantities and parameters are deterministic. However, there are many sources of uncertainties in this model. For example, the initial data of f and u come from experimental measurements, hence may have measurement errors. If one adopts the Maxwellian boundary condition for f with the accommodation coefficient, or the no-slip boundary condition for u against a wall with a nonzero velocity, then these boundary data will contain parameters, which come from direct measurements or matching with experimental data. Such parameters will also give rise to uncertainties. Furthermore, the parameters ϵ, κ, Re and the external field Φ come from measurements and have uncertainties. To provide reliable predictions and a guidance to improve the model, it is imperative to incorporate these uncertainties into the system, and quantify these uncertainties by numerically solving the resulting system with uncertain inputs.

In the last two decades, a large variety of numerical methods have been developed in the field of uncertainty quantification (UQ) [8,16,22,28,29]. Among these methods, the most popular ones are Monte-Carlo methods [23], stochastic collocation methods [2,4,24,30] and stochastic Galerkin methods [4,3]. The idea of Monte-Carlo methods is to sample randomly in the random space, which results in halfth order convergence. Stochastic collocation methods use sample points on a well-designed grid, and one can evaluate the statistical moments by numerical quadratures. Stochastic Galerkin methods start from an orthonormal basis in the random space, and approximate functions by truncated polynomial chaos expansions. By the Galerkin projection, a deterministic system of the expansion coefficients can be obtained. While Monte-Carlo methods have advantage in very high dimensional random spaces, the other two methods can achieve spectral accuracy if one adopts the generalized polynomial chaos (gPC) basis [31], which is a great advantage if the dimension of the random space is not too high. In this paper we focus on low dimensional random spaces, and adopt the stochastic Galerkin approach.

To effectively handle possible multiscales, where ϵ can be small or large, the AP approach has been proved to be very effective for (deterministic) kinetic problems (see [18,20]). For kinetic problems with uncertainties, the gPC based stochastic Galerkin (gPC-sG) method introduces a deterministic system which is basically a vector version of the deterministic counterparts, thus allowing one to utilize the deterministic AP framework to handle uncertain problems, in the sense of stochastic Asymptotic-Preserving (s-AP) [21]. A scheme is s-AP if the stochastic Galerkin (sG) method for the uncertain kinetic equation approaches to the sG method of the limiting (macroscopic) hydrodynamic equations as $\epsilon \to 0$. In this paper we adopt this approach, and propose an s-AP scheme for the kinetic-fluid model with uncertainties. In order to simplify the presentation and emphasize the main idea, we only consider the case of uncertain initial data. Uncertainties from other terms can be treated similarly in the sG framework, see [33].

Compared with the deterministic problem in [13], there are several new difficulties to overcome. First, the formal proof of the s-AP property is less obvious, due to the vector form of the scheme. Our proof is based on the observation that the gPC version of the FP operator $\vec{\mathcal{L}}$ is the deterministic FP operator $\mathcal{L}_0$ conjugated by a gPC version of the translation operator $\vec{\mathcal{T}}$ (see Section 3.3 for details). This observation gives rise to the equilibrium of $\vec{\mathcal{L}}$, and thus the hydrodynamic limit of the gPC system follows as $\epsilon \to 0$, which allows us to justify the s-AP property. Second, one needs to show that the resulted Helmholtz and Poisson systems, essential ingredients of the s-AP schemes, are well-defined systems. Indeed these properties, which are based on the positive-definiteness of the coefficient matrices in these systems, will be proven under reasonable and mild assumptions. Thirdly, to treat $\vec{\mathcal{L}}$ implicitly, which is needed for good numerical stability property, it is