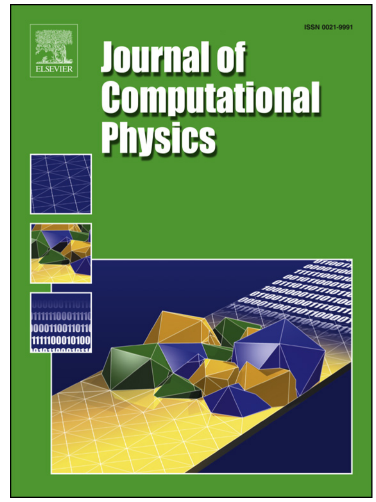

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Preconditioned Steepest Descent Methods for some Nonlinear Elliptic Equations Involving p-Laplacian Terms

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Abstract

We describe and analyze preconditioned steepest descent (PSD) solvers for fourth and sixth-order nonlinear elliptic equations that include p-Laplacian terms on periodic domains in 2 and 3 dimensions. The highest and lowest order terms of the equations are constant-coefficient, positive linear operators, which suggests a natural preconditioning strategy. Such nonlinear elliptic equations often arise from time discretization of parabolic equations that model various biological and physical phenomena, in particular, liquid crystals, thin film epitaxial growth and phase transformations. The analyses of the schemes involve the characterization of the strictly convex energies associated with the equations. We first give a general framework for PSD in Hilbert spaces. Based on certain reasonable assumptions of the linear pre-conditioner, a geometric convergence rate is shown for the nonlinear PSD iteration. We then apply the general theory to the fourth and sixth-order problems of interest, making use of Sobolev embedding and regularity results to confirm the appropriateness of our pre-conditioners for the regularized p-Lapacian problems. Our results include a sharper theoretical convergence result for p-Laplacian systems compared to what may be found in existing works. We demonstrate rigorously how to apply the theory in the finite dimensional setting using finite difference discretization methods. Numerical simulations for some important physical application problems – including thin film epitaxy with slope selection and the square phase field crystal model – are carried out to verify the efficiency of the scheme.

Keywords: Fourth-order nonlinear elliptic equation, sixth-order nonlinear elliptic equation, p-Laplacian operator, steepest descent, pre-conditioners, finite differences, Fast Fourier transform, thin film epitaxy, square phase field crystal model.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a rectangular domain. In this work we are interested in efficient solution techniques for fourth and sixth-order nonlinear elliptic equations that have p-Laplacian terms. The fourth-order problem reads as follows: given f Ω-periodic, find u Ω-periodic such that

$$u - s\nabla \cdot (|\nabla u|^{p-2}\nabla u) + s\varepsilon^2\Delta^2 u = f, \tag{1.1}$$

where $0 < \varepsilon \leq 1$ and s is a positive parameter. The sixth-order problem is as follows: given f, g Ω-periodic, find u, w Ω-periodic such that

$$u - \Delta w = g, \tag{1.2a}$$

$$s\lambda u - s\nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right) + s\varepsilon^2\Delta^2 u - w = f, \tag{1.2b}$$

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