Journal of Computational Physics 339 (2017) 1–30

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A sparsity regularization and total variation based computational framework for the inverse medium problem in scattering

Florian Bürgel [a,*], Kamil S. Kazimierski [b], Armin Lechleiter [a]

[a] Center for Industrial Mathematics, University of Bremen, Germany
[b] Institute of Mathematics and Scientific Computing,[1] University of Graz, Austria

ARTICLE INFO

Article history:
Received 5 August 2016
Received in revised form 3 February 2017
Accepted 5 March 2017
Available online 9 March 2017

Keywords:
Inverse scattering problem
Sparsity regularization
Total variation
Primal-dual algorithm

ABSTRACT

We present a fast computational framework for the inverse medium problem in scattering, i.e. we look at discretization, reconstruction and numerical performance. The Helmholtz equation in two and three dimensions is used as a physical model of scattering including point sources and plane waves as incident fields as well as near and far field measurements. For the reconstruction of the medium, we set up a rapid variational regularization scheme and indicate favorable choices of the various parameters. The underlying paradigm is, roughly speaking, to minimize the discrepancy between the reconstruction and measured data while, at the same time, taking into account various structural a-priori information via suitable penalty terms. In particular, the involved penalty terms are designed to promote information expected in real-world environments. To this end, a combination of sparsity promoting terms, total variation, and physical bounds of the inhomogeneous medium, e.g. positivity constraints, is employed in the regularization penalty. A primal-dual algorithm is used to solve the minimization problem related to the variational regularization. The computational feasibility, performance and efficiency of the proposed approach is demonstrated for synthetic as well as experimentally measured data.

1. Introduction

Inverse medium scattering problems seek to identify the refractive index of a penetrable medium from measurements of waves scattered from that medium. In acoustics, such parameter identification problems for instance are crucial for various non-destructive testing procedures based on ultrasound measurements. However, the numerical treatment of inverse medium scattering problems is challenging due to their intrinsic non-linearity and ill-posedness. Further, any inversion algorithm in addition needs to cope with huge system sizes arising after discretization. This is especially true for problems modeling all three space dimensions. In this paper, we propose a complete computational framework for inverse medium scattering at fixed frequency. The employed methods allow to take into account a-priori known structural properties of the searched-for refractive index in the variational reconstruction step via suitable penalty terms. Such properties for instance include (physical) bounds for the values of the refractive index, its sparsity in an a-priori known basis, or the presence of sharp edges.

* Corresponding author.
E-mail addresses: fbuergel@uni-bremen.de (F. Bürgel), kazimier@uni-graz.at (K.S. Kazimierski), lechleiter@math.uni-bremen.de (A. Lechleiter).
[1] The Institute of Mathematics and Scientific Computing is a member of NAWI Graz (www.nawigraz.at) and BioTechMed Graz (www.biotechmed.at).

http://dx.doi.org/10.1016/j.jcp.2017.03.011

Numerous algorithms solving inverse medium scattering problems are already established, all having specific advantages and disadvantages. A first class of algorithms (including ours) exploits Fréchet differentiability of the operator mapping the refractive index to the measured data. This allows to set up one of the various existing variants of regularized Newton-like inversion schemes that numerically show local convergence, see [6,14]. This class includes popular techniques as Kleinman and van den Berg's CG-based modified gradient method or Gutman and Klibanov's simplified gradient method. A second class of algorithms implements constructive uniqueness proofs for (features of) the refractive index, see, e.g. [9], which also includes so-called quantitative methods that merely identify spatial sets where the refractive index differs from its background values, see [6,15,16]. Finally, a third class of algorithms relies on a high or low frequency assumption to linearize the inversion problem in the corresponding asymptotic regime (e.g., physical optics approximation or Born approximation). This allows to use particular linear inversion methods for inverse medium scattering problems but remains of limited use if the wavelength is in the range of the size of the scattering object.

Whenever an inversion algorithm for the inverse medium scattering problem relies on multi-static data, see Fig. 1 below, then it is backed up by uniqueness results for the searched-for refractive index, both in two and three dimensions, see [12,5].

Our minimization-based reconstruction approach is known in inverse problems as (non-linear) Tikhonov regularization. For given near or far field measurements, we rely on the non-linear forward operator mapping the refractive index to the field measurements. As the refractive index n equals one outside the scatterer, we actually prefer to solve for the contrast $q := n^2 - 1$ of compact support, such that the forward operator $\mathcal{F}$ maps q to the measurements. Naturally, we tackle the inversion problem by seeking a contrast q such that $\mathcal{F}(q)$ matches measured data F_{meas}, i.e. $\|\mathcal{F}(q) - F_{\text{meas}}\|$ is small in some appropriate norm. To cope with ill-posedness (i.e., instability) of the inversion problem, this functional must be stabilized by adding a suitable penalty term, see [10]. Considering sparse refractive indices with respect to a pixel or wavelet basis, it is by now well-known that traditionally choosing the square of a Hilbert space norm yields worse results than choosing ℓ^p-norms for $p \in [1, 2)$ close to one, see [19] for a resulting algorithm based on soft-shrinkage iteration. We show in this paper that including further a-priori features of the refractive index further improves the reconstruction quality. This allows for instance to join total variation based with sparsity-promoting regularization. However, coping with additional penalties requires to use a more general minimization algorithm than the soft-shrinkage iteration used in [19]. To this end, we rely on a so-called primal-dual algorithm due to Pock, Bischof, Cremers and Chambolle, see [24,7], which offers enough flexibility for our setting while at the same time improving computation times considerably when compared with soft-shrinkage iterations. This algorithm is hence in particular applicable for three-dimensional inverse scattering problems; generally, its extensions to Banach spaces has been considered in [13], which also indicates numerical examples for one-dimensional deconvolution and two-dimensional phase reconstruction.

For any inverse problem dealing with parameter identification in differential equations, efficient inversion algorithms rely on an efficient solver for the underlying differential equation. For our setting, we describe solutions to the time-harmonic (direct) scattering problem via the so-called Lippmann–Schwinger equation. This volumetric integral equation can be efficiently discretized using a Fourier-based collocation scheme, which, in its original version, is due to Vainikko [29]. The resulting numerical scheme avoids several disadvantages of standard discretizations as, e.g., finite element or finite difference schemes: Numerical integration of singular functions or of the contrast is avoided, resulting solutions automatically satisfy the radiation condition, and the fast Fourier transform allows for fast matrix-vector multiplication. The resulting dense system matrix hence needs not to be set up, as the arising discrete system can be efficiently tackled by iterative techniques (we rely on the GMRES algorithm).

Of course, the inversion algorithm we propose depends on various parameters as, e.g., regularization and stopping parameters. All convergence theory for noise level tending to zero of such an iterative, Newton-like algorithm seems to require non-linearity conditions that are not yet verified. There is hence no parameter choice rule that theoretically guarantees convergence. Practically, we suggest suitable parameters that we determined in exhaustive numerical experiments.

We test feasibility and performance of our inversion algorithm via reconstructions from synthetic data in two and three dimensions as well as from experimentally measured data in two dimensions from Institute Fresnel, see [4]. Although the run-times should be taken with a pinch of salt because our computations rely on MATLAB, they give an impression of the performance of our algorithm. All numerical examples contained in this paper are computed by the MATLAB toolbox that is available on demand and contains both the mentioned inversion algorithms (primal-dual algorithm, soft-shrinkage iteration), as well as the mentioned integral equation solver for the direct scattering problem.

The rest of this paper is structured as follows: In Section 2 we give a brief introduction to direct and inverse scattering from inhomogeneous media. We tackle the discretization of the forward problem in Section 3 and the reconstruction in Section 4, where we develop a suitable penalty term, show explicitly how to apply the mentioned primal-dual algorithm, and introduce suitable stopping rules. We test our inversion algorithm with synthetic data in Section 5 in two and three dimensions. In Section 6 we apply the inversion algorithm to experimentally measured data.

Notation: We denote standard Lebesgue spaces with integrable pth power on a domain D by $L^p(D)$. The corresponding Sobolev spaces are $W^{s,p}(D)$ for $s \geq 0$ and $p \in [1, \infty]$; for $p = 2$, we abbreviate $H^s(D) = W^{s,2}(D)$. The notation of inner products is $\langle \cdot, \cdot \rangle$. In Section 3 vector and array quantities are denoted by an underlined symbol. To select an element in such quantities we employ a subindex or, where it is beneficial for readability, a bracket, i.e. $\underline{q}_j$ and $\underline{q}(j)$ both denote the jth element of $\underline{q}$. Sometimes we use the notation $f \cdot g$ to point out the pointwise multiplication of functions. The notation $(f \cdot)$ is used to denote the operator of pointwise multiplication (with a function f). In the same vein, $\underline{f} \odot \underline{g}$ and $(\underline{f} \odot)$ are used for element-wise multiplication of vector, matrix or array quantities as well as the related operator. Of course, for vectors