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A fully discrete, stable and conservative summation-by-parts formulation for deforming interfaces

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ABSTRACT

We introduce an interface/coupling procedure for hyperbolic problems posed on time-dependent curved multi-domains. First, we transform the problem from Cartesian to boundary-conforming curvilinear coordinates and apply the energy method to derive well-posed and conservative interface conditions.
Next, we discretize the problem in space and time by employing finite difference operators that satisfy a summation-by-parts rule. The interface condition is imposed weakly using a penalty formulation. We show how to formulate the penalty operators such that the coupling procedure is automatically adjusted to the movements and deformations of the interface, while both stability and conservation conditions are respected.
The developed techniques are illustrated by performing numerical experiments on the linearized Euler equations and the Maxwell equations. The results corroborate the stability and accuracy of the fully discrete approximations.

1. Introduction

Multi-block schemes and in particular interface procedures that use Summation-by-Parts (SBP) operators together with the Simultaneous Approximation Term (SAT) technique [2], have previously been investigated in terms of conservation, stability and accuracy [3,5,6,10,11,26–29]. The focus of the SBP-SAT methodology has been, so far, mostly on time-independent spatial domains with a notable exception being [1].

In this article, we extend the techniques introduced in [1] for handling time-dependent boundaries in a single domain problem, to a multi-domain context with deforming interfaces. The new time-dependent interface formulation is conservative, provably stable and high order accurate.

The rest of this article proceeds as follows. We start, in section 2, by transforming the continuous problem from Cartesian to curvilinear coordinates. Next, we study the problem using the energy method, our analytical tool, and derive conditions for conservation and well-posedness. Section 3 deals with the discrete problem where we study conservation and stability of the interface procedures and show the similarities with the continuous problem. In section 4, numerical experiments are performed to show the accuracy and the usefulness of the scheme. Finally, we summarize and draw conclusions in section 5.

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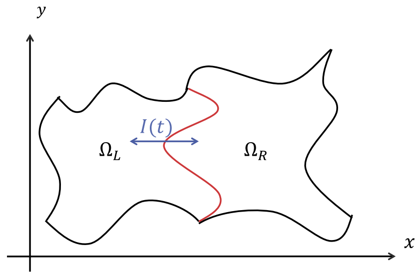

Fig. 1. A schematic of the domains $\Omega_{L,R}(t)$ and the time-dependent interface $I(t)$.

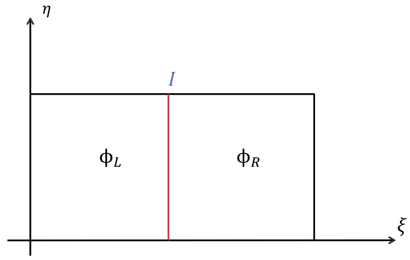

Fig. 2. A schematic of the transformed domains $\Phi_{L,R}$ and the time-independent interface I.

2. The continuous problem

Consider one hyperbolic problem with solution W posed on two nearby spatial domains, as

$$\begin{aligned} U_t + \hat{A}U_x + \hat{B}U_y &= 0, \quad (x,\ y) \in \Omega_L(t), \quad t \in [0,\ T], \\ V_t + \hat{A}V_x + \hat{B}V_y &= 0, \quad (x,\ y) \in \Omega_R(t), \quad t \in [0,\ T]. \end{aligned} \tag{1}$$

The solutions U, V represent the left L and right R values of the continuous solution W posed on the union of $\Omega_L(t)$ and $\Omega_R(t)$, where $\Omega_{L,R}(t)$ are the time-dependent sub-domains. In (1), x and y are the spatial coordinates and t represents time. The matrices $\hat{A}$ and $\hat{B}$ are constant, symmetric [19,25] and of size $l \times l$. We focus on the case where the deformations of $\Omega_{L,R}(t)$ are mainly caused by the moving and/or deforming interface $I(t)$, see Fig. 1.

Next, two time-dependent invertible Lagrangian–Eulerian transformations [22] of $\Omega_{L,R}(t)$ from Cartesian to curvilinear coordinates as

$$x(\tau,\xi,\eta) \rightleftharpoons \xi(t,x,y), \quad y(\tau,\xi,\eta) \rightleftharpoons \eta(t,x,\ y), \quad \tau = t, \tag{2}$$

are introduced. We consider boundary-conforming curvilinear coordinates where the boundaries of $\Omega_{L,R}$ are composed of segments with constant ξ,η, resulting in fixed spatial sub-domains after the transformations [30]. The fixed sub-domains are denoted by $\Phi_{L,R}$ and shown schematically in Fig. 2. The interface between Φ_L and Φ_R, denoted by I, is now time-independent in η,ξ space.

The transformations we have used both satisfy

$$\begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \\ \partial/\partial\tau \end{bmatrix} = \underbrace{\begin{bmatrix} x_\xi & y_\xi & 0 \\ x_\eta & y_\eta & 0 \\ x_\tau & y_\tau & 1 \end{bmatrix}}_{:=[J]} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial t \end{bmatrix}, \tag{3}$$

where the subscripts ξ,η and τ denote partial derivatives and $[J]$ is the Jacobian matrix of the transformation. By considering the Jacobian matrix of the inverse transformation, the following metric relations are obtained [12,30]

$$\begin{aligned} J\xi_t &= x_\eta y_\tau - x_\tau y_\eta, \quad J\xi_x = y_\eta, \quad\ \ J\xi_y = -x_\eta, \\ J\eta_t &= y_\xi x_\tau - x_\xi y_\tau, \quad J\eta_x = -y_\xi, \quad J\eta_y = x_\xi, \end{aligned} \tag{4}$$

where $J = x_\xi y_\eta - x_\eta y_\xi > 0$ is the determinant of $[J]$.

For non-singular (invertible) transformations, the Geometric Conservation Law (GCL) [12,30] holds, i.e.

$$\begin{aligned} J_\tau + (J\xi_t)_\xi + (J\eta_t)_\eta &= 0, \\ (J\xi_x)_\xi + (J\eta_x)_\eta &= 0, \\ (J\xi_y)_\xi + (J\eta_y)_\eta &= 0. \end{aligned} \tag{5}$$

Remark 1. *For ease of presentation, we have not distinguished between the left and right transformations in (2), (3), (4) and (5) whereas in the remainder of the article, we will show this by using the subscripts L and R.*

Next, the governing equations in (1) are expressed in terms of ξ, η and τ by using the chain rule and multiplying the results with $J_{L,R}$, as

$$\begin{aligned} J_L U_\tau + A_L U_\xi + B_L U_\eta &= 0, \quad (\xi,\ \eta) \in \Phi_L, \quad \tau \in [0,\ T], \\ J_R V_\tau + A_R V_\xi + B_R V_\eta &= 0, \quad (\xi,\ \eta) \in \Phi_R, \quad \tau \in [0,\ T], \end{aligned} \tag{6}$$

where