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Constructing non-reflecting boundary conditions using summation-by-parts in time

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ABSTRACT

In this paper we provide a new approach for constructing non-reflecting boundary conditions. The boundary conditions are based on summation-by-parts operators and derived without Laplace transformation in time. We prove that the new non-reflecting boundary conditions yield a well-posed problem and that the corresponding numerical approximation is unconditionally stable. The analysis is demonstrated on a hyperbolic system in two space dimensions, and the theoretical results are confirmed by numerical experiments.

1. Introduction

Many applications in physics and engineering involve unbounded physical domains, which must be limited by Artificial Boundary Conditions (ABC's). These boundary conditions will, if not chosen appropriately, produce non-physical reflections that pollute the solution in the interior of the domain.

Non-Reflecting Boundary Conditions (NRBC's), i.e. ABC's that do not produce reflections, are in most cases obtained in a transformed dual space, see [4,5,15]. The NRBC's are exact in the transformed space, but hard to implement since they are expressed in the dual variables. To circumvent this issue, the exact boundary conditions are often approximated using various types of expansions in combination with suitable size assumptions on the frequencies involved [4,15]. The resulting approximate boundary conditions are local in both space and time, and relatively easy to implement.

Unfortunately, this approach has some drawbacks. First and foremost, although the exact NRBC's result in a well-posed problem, an approximation of these often leads to an ill-posed problem [4], and consequently an unstable scheme. Secondly, even if the approximate NRBC's lead to a well-posed problem and a stable scheme, the accuracy is ruined since the amplitude of the reflections is independent of the mesh size. The reflections will therefore not vanish during mesh-refinement [5].

Another quite different approach for constructing approximate NRBC's is to introduce buffer zones as artificial boundaries, where incoming waves are damped. When the interface between the computational domain and the buffer zone is exactly non-reflecting, it is called a Perfectly Matched Layer (PML) [1]. Buffer zone techniques will not be discussed further in this paper. For comprehensive reviews of NRBC's including PML's as well as other techniques, the reader is referred to [9,6,18].

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The use of Summation-By-Parts (SBP) operators [7,11,16] for the discrete formulation of partial differential equations has proven to be very successful. The boundary conditions for SBP approximations are implemented weakly by using Simultaneous Approximation Terms (SAT) [2,3]. In this paper, we will make use of the SBP-SAT technique in time [10,12,13] when constructing the NRBC's. By using this technique, the complete analysis can be performed in real space, and hence, the NRBC's can be implemented relatively easily. The resulting boundary conditions are global in space and time, and the whole procedure bypasses the transformation and accuracy problems mentioned above.

The remainder of this paper will proceed as follows. We describe the SBP-SAT technique on a one-dimensional problem in Section 2. In Section 3, NRBC's are derived for a general two-dimensional hyperbolic problem. It is shown that the boundary conditions result in a well-posed problem. By considering the discrete problem, we introduce an alternative way of deriving the NRBC's in Section 4. The theoretical results are verified by numerical experiments in Section 5. Finally, in Section 6, we summarize the results and draw conclusions.

2. The SBP-SAT technique

To introduce the SBP-SAT technique, we consider the advection equation in one space dimension,

$$\begin{aligned} &u_t + u_x = 0, \quad x \in [0,1], \quad t > 0, \\ &u(0,t) = g(t), \\ &u(x,0) = f(x). \end{aligned} \tag{1}$$

To examine well-posedness of (1), we multiply with u and integrate in space and time to get

$$||u||^2 + \int_0^t u^2(1,\tau)d\tau = ||f||^2 + \int_0^t g^2(\tau)d\tau, \tag{2}$$

where the boundary and initial conditions in (1) and the notation $||u||^2 = \int_0^1 u^2(x,t)dx$ have been used. Obviously, u is bounded by the data f and g which implies that the problem is strongly well-posed. For a detailed discussion on well-posedness of initial boundary value problems, see [8,14].

2.1. The fully discrete problem

Equation (1) is discretized using the SBP-SAT technique in space and time,

$$(P_t^{-1}Q_t \otimes I_x)v + (I_t \otimes P_x^{-1}Q_x)v = \alpha_t(P_t^{-1}E_{0t} \otimes I_x)(v - \bar{f}) + \alpha_x(I_t \otimes P_x^{-1}E_{0x})(v - \bar{g}), \tag{3}$$

where $P_{t,x}$ and $Q_{t,x}$ are the SBP-operators satisfying $P_{t,x} = P_{t,x}^T > 0$ and $Q_{t,x} + Q_{t,x}^T = diag(-1, 0, ..., 0, 1)$. Moreover, P_t is proportional to the time step Δt and P_x is proportional to the grid spacing Δx. In (3), $E_{0t,x} = diag(1, 0, ..., 0)$ are of the same sizes as $P_{t,x}$, respectively, and $\bar{f}$, $\bar{g}$ are grid functions with the values of f, g injected at appropriate grid points. The symbol $\otimes$ denotes the Kronecker product which, for two arbitrary matrices M and N, is defined as

$$M \otimes N = \begin{bmatrix} M_{11}N & \dots & \dots & M_{1n}N \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ M_{m1}N & \dots & \dots & M_{mn}N \end{bmatrix}.$$

The penalty coefficients $\alpha_{t,x}$ will be determined such that the scheme becomes stable.

We now multiply (3) with $v^T(P_t \otimes P_x)$ from the left, choose $\alpha_t = \alpha_x = -1$ and add the transpose of the outcome to itself to find

$$\begin{aligned} v^T(E_{Nt} \otimes P_x)v + v^T(P_t \otimes E_{Nx})v &= \bar{f}^T(E_{0t} \otimes P_x)\bar{f} + \bar{g}^T(P_t \otimes E_{0x})\bar{g} \\ &\quad -(v - \bar{f})^T(E_{0t} \otimes P_x)(v - \bar{f}) - (v - \bar{g})^T(P_t \otimes E_{0x})(v - \bar{g}). \end{aligned} \tag{4}$$

Since the matrices $E_{0x,t}$ and $E_{Nx,t}$ in (4) are positive semi-definite, v is bounded by the data $\bar{f}, \bar{g}$, just as in the continuous case. This implies that the numerical scheme is strongly stable. The estimate (4) is valid for any choice of timestep and hence (3) is unconditionally stable.

Furthermore, the terms on the left-hand side of (4) mimic the terms on the left-hand side of (2). Also, if $v = \bar{g}$ at $x = 0$ and $v = \bar{f}$ when $t = 0$, the right-hand side of (4) mimics the right-hand side of (2). In summary: the discrete energy estimate (4) mimics the continuous energy estimate (2) with additional damping terms that vanish with mesh-refinement. For more details on the SBP-SAT technique, the reader is referred to [17].