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# A fast immersed boundary method for external incompressible viscous flows using lattice Green's functions

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ABSTRACT

A new parallel, computationally efficient immersed boundary method for solving three-dimensional, viscous, incompressible flows on unbounded domains is presented. Immersed surfaces with prescribed motions are generated using the interpolation and regularization operators obtained from the discrete delta function approach of the original (Peskin's) immersed boundary method. Unlike Peskin's method, boundary forces are regarded as Lagrange multipliers that are used to satisfy the no-slip condition. The incompressible Navier–Stokes equations are discretized on an unbounded staggered Cartesian grid and are solved in a finite number of operations using lattice Green's function techniques. These techniques are used to automatically enforce the natural free-space boundary conditions and to implement a novel block-wise adaptive grid that significantly reduces the run-time cost of solutions by limiting operations to grid cells in the immediate vicinity and near-wake region of the immersed surface. These techniques also enable the construction of practical discrete viscous integrating factors that are used in combination with specialized half-explicit Runge–Kutta schemes to accurately and efficiently solve the differential algebraic equations describing the discrete momentum equation, incompressibility constraint, and no-slip constraint. Linear systems of equations resulting from the time integration scheme are efficiently solved using an approximation-free nested projection technique. The algebraic properties of the discrete operators are used to reduce projection steps to simple discrete elliptic problems, e.g. discrete Poisson problems, that are compatible with recent parallel fast multipole methods for difference equations. Numerical experiments on low-aspect-ratio flat plates and spheres at Reynolds numbers up to 3700 are used to verify the accuracy and physical fidelity of the formulation.



## 1. Introduction

Immersed boundary (IB) methods are numerical techniques for solving PDEs on Eulerian grids with immersed surfaces that are described by Lagrangian structures [1–3]. Immersed surfaces are emulated without modifying the underlying PDE discretization by the addition of forcing terms and constraint equations resulting from the regularization of Dirac delta convolutions linking Eulerian and Lagrangian quantities. In addition to circumventing computationally expensive body-fitted grid generation, this approach facilitates the extensions of robust and efficient solvers, e.g. Cartesian-grid methods, to problems involving immersed surfaces. The original IB method [4] was developed for flexible elastic structures, but has since

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been extended to handle more general fluid–structure interactions, including rigid bodies and bodies with prescribed motions [5–13]. The numerous variants of the IB method and some of their higher-order extensions are reviewed in [1–3]. Here, we focus on *distributed Lagrange multiplier (DLM) methods* [7,9–12,14–16] since they are particularly robust IB methods for computing flows around bodies with prescribed motions [3].

DLM methods treat boundary forces as Lagrange multipliers used to enforce prescribed surface boundary conditions. For the case of fluid flows, these methods are typically expressible in forms analogous to traditional fractional-step and projection methods and can be distinguished in part by differences in splitting errors, underlying PDEs, discretization schemes, and numerical solvers [3,9,12]. The null-space (discrete streamfunction) projection approach [10] and the Rigid-IBAMR solver [12] are examples of robust incompressible Navier–Stokes DLM methods free of splitting errors. The absence of splitting errors ensures that solutions retain the accuracy, stability, and physical fidelity of the PDE discretization scheme [9,10,12,17,18].

IB methods for external flows typically employ spatially truncated fluid domains with approximate free-space boundary conditions, which in turn introduce *blockage* errors that adversely affect the accuracy and can even change the dynamics of the numerical solution [19–22]. Large computational domains in combination with stretched grids [9,23,24], local grid refinement [12,25,26], and far-field approximation techniques [10] are commonly used to reduce blockage errors. These techniques often increase the number of computational elements and require the use of numerical solvers that are less efficient than regular-grid solvers (e.g. FFT techniques, multigrid, etc.). Additionally, these techniques typically make use of discretization schemes that do not formally share the same conservation, commutativity, orthogonality, and symmetry properties of standard staggered Cartesian discretizations of infinite (periodic or unbounded) domains.

In order to eliminate the errors associated with artificial boundary conditions and to limit operations to small regions dictating the flow evolution (e.g. regions of significant vorticity), while preserving the efficiency and robustness inherent to Cartesian staggered grid methods, we proposed [27] a fast incompressible Navier–Stokes solver based on the fundamental solution, or lattice Green's function (LGF), of discrete operators. Similar to particle and vortex methods, LGF techniques have efficient nodal distributions, automatically enforce natural free-space boundary conditions, and can be evaluated using fast multipole methods (FMMs), e.g. the 2D serial method [28] and the 3D parallel method [29]. Using the LGF-FMM [29] in combination with an projection technique that is free of splitting errors, the LGF flow solver [27] computes fast, parallel solutions to the viscous integrating factor (IF) half-explicit Runge–Kutta (HERK) time integration scheme used to solve the velocity and pressure of the flow.

The present method shares common features with the rigid-body steady Stokes solver of Bringley and Peskin [11]—both are IB-DLM formulations on unbounded Cartesian grids, and both use LGFs to solve elliptic difference equations associated with the immersed surface. However, differences in problem definition (steady Stokes equations versus unsteady incompressible Navier–Stokes equations) result in methods that differ significantly in their formulation and solution techniques. The discussion of [11] states that "using an unbounded domain is no longer possible" for the case of the Navier–Stokes equations due to complications in force computations arising from the presence of the non-linear term, the inability to eliminate fluid variables, and prohibitively high computational costs. In contrast, we demonstrate here that by coupling LGFs with suitable spatial discretizations, time integrators, fast solvers, and adaptive grid techniques, it is possible to efficiently compute unsteady incompressible flows on unbounded domains.

The present method numerically solves the IB formulation for the incompressible Navier–Stokes equations given, in its continuous form, by

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u} + \int_{\Gamma(t)} \mathbf{f}_\Gamma(\boldsymbol{\xi},t)\,\delta\left(\mathbf{X}(\boldsymbol{\xi},t)-\mathbf{x}\right)d\boldsymbol{\xi}, \tag{1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1b}$$

$$\int_{\mathbb{R}^3} \mathbf{u}(\mathbf{x},t)\,\delta\left(\mathbf{x}-\mathbf{X}(\boldsymbol{\xi},t)\right)d\mathbf{x} = \mathbf{u}_\Gamma(\boldsymbol{\xi},t), \tag{1c}$$

where the immersed surface $\Gamma(t)$ is parametrized by $\boldsymbol{\xi}$, and $\mathbf{X}(\boldsymbol{\xi},t)\in\Gamma(t)$. The velocity, pressure, and Reynolds number of the flow are denoted by $\mathbf{u}(\mathbf{x},t)$, $p(\mathbf{x},t)$, and Re. Here, Eq. (1c) is taken to be the no-slip condition on $\Gamma(t)$, where $\mathbf{u}_\Gamma(\boldsymbol{\xi},t) = [\partial\mathbf{X}/\partial t](\boldsymbol{\xi},t)$.[1] The body force term in Eq. (1a), with unknown force density $\mathbf{f}_\Gamma(\boldsymbol{\xi},t)$, is computed so that $\mathbf{u}(\mathbf{x},t)$ satisfies Eq. (1c). The fluid variables $\mathbf{u}(\mathbf{x},t)$ and $p(\mathbf{x},t)$ are defined for all $\mathbf{x}\in\mathbb{R}^3$, and subject to the boundary condition $\mathbf{u}(\mathbf{x},t)\to 0$ as $|\mathbf{x}|\to\infty$.

In this paper, we extend the unbounded domain LGF flow solver [27] to include immersed surfaces with prescribed motions using a Lagrange multiplier approach and an accelerating reference frame formulation. In Section 2, we provide an overview of the discretization of Eq. (1) on unbounded fluid domains, and extend the LGF techniques and IF-HERK time integration schemes of [27] to efficiently and accurately include immersed boundaries. Linear systems of equations arising

[1] For the case of closed immersed surfaces, we limit our attention to prescribed motions that are volume conserving or, equivalently, surface velocities that satisfy the incompressibility condition $\int_{\Gamma(t)} \mathbf{u}_\Gamma(\boldsymbol{\xi},t)\cdot\hat{\mathbf{n}}(\boldsymbol{\xi},t)\,d\boldsymbol{\xi} = 0$, where $\hat{\mathbf{n}}$ is the surface normal unit vector.