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An efficient implementation of a high-order filter for a cubed-sphere spectral element model

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A B S T R A C T

A parallel-scalable, isotropic, scale-selective spatial filter was developed for the cubed-sphere spectral element model on the sphere. The filter equation is a high-order elliptic (Helmholtz) equation based on the spherical Laplacian operator, which is transformed into cubed-sphere local coordinates. The Laplacian operator is discretized on the computational domain, i.e., on each cell, by the spectral element method with Gauss–Lobatto Lagrange interpolating polynomials (GLLIPs) as the orthogonal basis functions. On the global domain, the discrete filter equation yielded a linear system represented by a highly sparse matrix. The density of this matrix increases quadratically (linearly) with the order of GLLIP (order of the filter), and the linear system is solved in only $O(N_g)$ operations, where N_g is the total number of grid points. The solution, obtained by a row reduction method, demonstrated the typical accuracy and convergence rate of the cubed-sphere spectral element method. To achieve computational efficiency on parallel computers, the linear system was treated by an inverse matrix method (a sparse matrix–vector multiplication). The density of the inverse matrix was lowered to only a few times of the original sparse matrix without degrading the accuracy of the solution. For better computational efficiency, a local-domain high-order filter was introduced: The filter equation is applied to multiple cells, and then the central cell was only used to reconstruct the filtered field. The parallel efficiency of applying the inverse matrix method to the global- and local-domain filter was evaluated by the scalability on a distributed-memory parallel computer. The scale-selective performance of the filter was demonstrated on Earth topography. The usefulness of the filter as a hyper-viscosity for the vorticity equation was also demonstrated.

1. Introduction

The cubed-sphere grid consists of six identical domains, which are generated by projecting the inscribed cube's grid onto the spherical surface. The cubed-sphere grid has emerged as a promising alternative to the traditional latitude–longitude (lat–lon) grid as it provides almost uniform resolution on the globe, and is now widely accepted in numerical models of geophysical problems [1,2,5–7,9,11–13,17–23]. A variety of numerical methods have been successfully adapted to the cubed-sphere grid [5,6,11,13,17–19,22]. Unlike the lat–lon grid, where the grid-points align along the parallel and the meridian, the cubed-sphere grid discretizes the governing equations on locally defined computational domains (or cells). In the discretization process, the discontinuities along the edges of the cube and the boundaries of the elements within the cube's

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face are minimized by various connection strategies. This discontinuity problem, or its associated grid-imprinting, becomes more serious when discretizing higher-order differential equations such as the Laplacian operator [17,24,26].

Among the numerical methods for the cubed-sphere grid is the spectral element method (SEM), which retains the advantages of the finite element and spectral methods [2,6,17,18]. SEM provides spectral accuracy without requiring the spectral transform, because the grid point values are handled by spatial differentiation and time stepping. Like other models, SEM is affected by numerical noise introduced by the nonlinear terms. However, this noise can be controlled to a certain extent by introducing scale-selective viscosity, which is represented by the Laplacian operator (i.e., a harmonic viscosity). Unfortunately, under the Courant–Friedrichs–Lewy (CFL) condition, applying the Laplacian-type high-order viscosity in cubed-sphere SEM restricts the viscosity coefficient [5]. Specifically, when each face of the cube is divided into $N_e \times N_e$ cells, the largest absolute-value eigenvalue of the Laplacian operator for the cubed-sphere SEM ($|\alpha|_{\max}$, where all eigenvalues are negative) on a unit sphere is quadratically and quartically proportional to N_e and the order of the GLLIPs N, respectively:

$$|\alpha|_{\max} \approx \begin{cases} 1.51 \times (N+0.5)^4 N_e^2 & \text{for equidistance projection} \\ 0.579 \times (N+0.63)^4 N_e^2 & \text{for equiangular projection} \end{cases} . \tag{i-1}$$

The first (second) term in (i-1) refers to equidistant (equiangular) gnomonic projection of the sphere onto the cube. If the hyperdiffusion (or hyperviscosity) equation for an arbitrary variable T, $\partial T/\partial t = (-1)^{q+1}\kappa\nabla^{2q}T$, is solved by simple time-differencing, where κ and q are the viscosity coefficient and the order of the Laplacian, respectively, the restriction on the viscosity coefficient is easily observed as

$$\gamma \equiv \kappa\Delta t \le \left[|\alpha|_{\max}^q\right]^{-1}, \tag{i-2}$$

where Δt is the time step of the time-differencing method.

Although an implicit treatment will remove the restriction on the viscosity coefficient (or time step) in Laplacian-type high-order viscosity equations [4], it incurs additional computational cost for solving the elliptic equations. In terms of number of operations, the computational cost in this case is not severe. More important is the availability of an efficient parallel algorithm; that is, an algorithm that minimizes both the number of communications (latency) and the communication size (bandwidth). An elliptic equation solver based on the row reduction method [4,10] performs forward or backward substitutions, which inevitably exchange the values between pairs of far-distant grid points on different elements. Therefore, to achieve an accurate SEM with high parallel efficiency, the elliptic equation must be solved by a method other than the row reduction method. Once the filter is set up, it is repeatedly applied over different sets of input data [4,6,7,9], particularly during the time-stepping procedure. Therefore, the most promising alternative to row reduction is the matrix operation method, which performs by forward operation of the matrix, enabling much easier parallel implementation than the row reduction method. It should be noted that inverting the Laplacian-related matrix is unavoidable; fortunately, the matrix operation method requires only a single inversion. The inversion problem may be overcome by a local-domain filter, which incorporates only a limited number of cells. In this case, however, the number of cells must be carefully chosen to preserve the accuracy and efficiency of the solutions.

The present study investigates an elliptic solver for the cubed-sphere SEM, which constitutes a high-order Laplacian-type filter on the sphere. The spherical Laplacian operator in the filter equation is discretized using GLLIPs on the cubed-sphere grid. The remainder of this paper is organized as follows. Section 2 presents the elliptic equation and discretization methods on the cubed-sphere grid, and discusses the computational efficiency and accuracy of two solution methods for discrete elliptic equations. Section 3 examines the property of the filter in terms of the weighted average. Section 4 determines the parallel efficiency of our proposed methods in solving a huge linear system associated with the elliptic equation. Section 5 improves the computational efficiency by replacing the global-domain filter with a local-domain filter. The performance of the filter, as a smoothing operator for global grid data and a hyperviscosity in the time-dependent vorticity equation, is evaluated in Section 6. A summary is given in Section 7.

2. SEM discrete elliptic equations on a cubed-sphere grid

2.1. Spherical Laplacian operator and discretization

The spherical Laplacian operator on the unit sphere is written as

$$\nabla^2 \equiv \frac{1}{(1-z^2)}\frac{\partial^2}{\partial\lambda^2} + \frac{\partial}{\partial z}\left(1-z^2\right)\frac{\partial}{\partial z}, \tag{1}$$

where λ and z are the longitude and the sine of latitude, respectively. The Laplacian operator (1) can be transformed into various forms depending on the grid system. The cubed-sphere grid is an orthogonal grid generated by projecting the spherical surface onto an inscribed cube, as explained in [17,18]. The six-faced cubed-sphere grid is divided into cell elements, on which a local grid is defined. In the local 2-dimensional cell on the cubed-sphere grid, (1) is rewritten as

$$\nabla^2 = \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^1}\left[\sqrt{G}G^{11}\frac{\partial}{\partial x^1} + \sqrt{G}G^{12}\frac{\partial}{\partial x^2}\right] + \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^2}\left[\sqrt{G}G^{21}\frac{\partial}{\partial x^1} + \sqrt{G}G^{22}\frac{\partial}{\partial x^2}\right], \tag{2}$$