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# A fast and well-conditioned spectral method for singular integral equations

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ABSTRACT

We develop a spectral method for solving univariate singular integral equations over unions of intervals by utilizing Chebyshev and ultraspherical polynomials to reformulate the equations as almost-banded infinite-dimensional systems. This is accomplished by utilizing low rank approximations for sparse representations of the bivariate kernels. The resulting system can be solved in $\mathcal{O}(m^2n)$ operations using an adaptive QR factorization, where $m$ is the bandwidth and $n$ is the optimal number of unknowns needed to resolve the true solution. The complexity is reduced to $\mathcal{O}(mn)$ operations by pre-caching the QR factorization when the same operator is used for multiple right-hand sides. Stability is proved by showing that the resulting linear operator can be diagonally preconditioned to be a compact perturbation of the identity. Applications considered include the Faraday cage, and acoustic scattering for the Helmholtz and gravity Helmholtz equations, including spectrally accurate numerical evaluation of the far- and near-field solution. The JULIA software package `SingularIntegralEquations.jl` implements our method with a convenient, user-friendly interface.



## 1. Introduction

Singular integral equations are prevalent in the study of fracture mechanics [1], acoustic scattering problems [2–6], Stokes flow [7], Riemann–Hilbert problems [8], and beam physics [9,10]. We develop a fast and stable algorithm for the solution of univariate singular integral equations of general form [11]

$$\underset{\Gamma}{⨍\!\!\!\!\times} K(x,y)u(y)\,\mathrm{d}y = f(x), \quad \text{for} \quad x \in \Gamma, \qquad \mathcal{B}u = \mathbf{c}, \tag{1}$$

where $K(x,y)$ is singular along the line $y = x$, the $\times$ in the integral sign denotes either the Cauchy principal value or the Hadamard finite-part, $\Gamma$ is a union of bounded smooth open arcs in $\mathbb{R}^2$, and $\mathcal{B}$ is a list of functionals. To be precise, we consider the prototypical singular integral equations on $[-1,1]$ given by:

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$$\frac{1}{\pi}\fint_{-1}^{1}\left(\frac{K_1(x,y)}{(y-x)^2}+\frac{K_2(x,y)}{y-x}+\log|y-x|K_3(x,y)+K_4(x,y)\right)u(y)\,\mathrm{d}y=f(x),\quad \text{for}\quad x\in[-1,1],$$

where $K_1,\ldots,K_4$ are continuous bivariate kernels.

In this work, we use several remarkable properties of Chebyshev polynomials including their spectral convergence, explicit formulæ for their Hilbert and Cauchy transforms, and low rank bivariate approximations to construct a fast and well-conditioned spectral method for solving univariate singular integral equations. Chebyshev and ultraspherical polynomials are utilized to convert singular integral operators into numerically banded infinite-dimensional operators. To represent bivariate kernels, we use the low rank approximations of [12], where expansions in Chebyshev polynomials are constructed via sums of outer products of univariate Chebyshev expansions. The minimal solution to the recurrence relation is automatically revealed by the adaptive QR factorization of [13]. Diagonal right preconditioners are derived for integral equations encoding Dirichlet and Neumann boundary conditions such that the preconditioned operators are compact perturbations of the identity.

The inspiration behind the proposed numerical method is the ultraspherical spectral method for solving ordinary differential equations [13], where ordinary differential equations are converted to infinite-dimensional almost banded linear systems (an almost banded operator is a banded operator apart from a finite number of dense rows). These systems can be solved in infinite-dimensions, i.e., without truncating the operators [14], as implemented in `ApproxFun.jl` [15] in the Julia programming language [16,17]. The Julia software package `SingularIntegralEquations.jl` [18] implements our method with a convenient, user-friendly interface. As an extension of this framework for infinite-dimensional linear algebra, mixed equations involving derivatives and singular integral operators can be solved in a unified way.

Several classical numerical methods exist for singular Fredholm integral equations of the first kind. These include: the Nyström method [19–21], whereby integral operators are approximated by quadrature rules; the collocation method [22,23], where approximate solutions in a finite-dimensional subspace are required to satisfy the integral equation at a finite number of collocation points; and the Galerkin method [24,25], where the approximate solution is sought from an orthogonal subspace and is minimal in the energy norm. The use of hybrid Gauss-trapezoidal quadrature rules [26–29] can significantly increase the convergence rates when treating weakly singular kernels.

Numerous methods have exploited the underlying structure of the linear systems arising from discretizing integral equations. The most celebrated of these is the Fast Multipole Method of Greengard and Rokhlin [30]. Other characterizations in terms of semi-separability or other hierarchies have also gained prominence [31–33]. Exploiting the matrix structure allows for fast matrix–vector products, which then allows for Krylov subspace methods [34] to be extremely competitive. For scattering of the Helmholtz equation in very special geometries, hybrid numerical-asymptotic methods have been derived for frequency-independent solutions to the Dirichlet and Neumann problems [4,35,6,36].

Previous works on Chebyshev-based methods for singular integral equations include Frenkel [37], which derives recurrence relations for the Chebyshev expansion of a singular integral equation after expanding the bivariate kernel in a basis of Chebyshev polynomials of the first kind in both variables, and Chan et al. [38,39] in fracture mechanics, among others. A similar analysis in [40] is used for hypersingular integrodifferential equations by expanding the bivariate kernel in a basis of Chebyshev polynomials of the second kind. This paper is an extension of these ideas with essential practical numerical considerations.

**Remarks.**

1. Combined with fast multiplication of Chebyshev series, our method is suitable for use in iterative Krylov subspace methods.
2. There is a great diversity of integral equation formulations. The choice of formulation depends on many properties, including for example, whether the boundary is open or closed and whether there are resonances. Most equations involve operators that contain manipulations of the fundamental solution, which would still satisfy the requirements of our method. However, we focus on the direct integral equations to retain a simple exposition.

## 2. Boundary integral equations in two dimensions

In two dimensions, let $\mathbf{x}=(x_1,x_2)$ and $\mathbf{y}=(y_1,y_2)$. Positive definite second-order linear elliptic partial differential operators (PDOs) with variable coefficients are always reducible to the following *canonical form* [41]:

$$\mathbf{L}\{u\}=\Delta u+a\frac{\partial u}{\partial x_1}+b\frac{\partial u}{\partial x_2}+cu. \tag{2}$$

Let $\Phi(\mathbf{x},\mathbf{y})$ denote the positive definite *fundamental solution* of (2) satisfying the formal partial differential equation (PDE)