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Adaptive multiresolution semi-Lagrangian discontinuous Galerkin methods for the Vlasov equations

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ABSTRACT

We develop adaptive numerical schemes for the Vlasov equation by combining discontinuous Galerkin discretisation, multiresolution analysis and semi-Lagrangian time integration. We implement a tree based structure in order to achieve adaptivity. Both multi-wavelets and discontinuous Galerkin rely on a local polynomial basis. The schemes are tested and validated using Vlasov–Poisson equations for plasma physics and astrophysics.

1. Introduction

Many numerical experiments are performed on the Vlasov–Poisson problem since it is a well known (but not fully understood) system from plasma physics and a major issue for future simulation of large scale plasmas. Our goal is to develop adaptive numerical schemes using discontinuous Galerkin discretisation combined with semi-Lagrangian description whose mesh refinement is based on multi-wavelets. In [16], the authors start from the standard formulation of finite-volume schemes for conservation laws and transform them into adaptive finite-volume schemes by using multiscale representations of the data. In the same philosophy, starting from the standard formulation of the discontinuous Galerkin method for conservation laws the authors of Refs. [3,38,28,32,27] use a multi-wavelet representation of the unknowns to put these schemes into an adaptive setting. Following the spirit of these works, and in order to solve the transport and continuity equations, we use a multi-wavelet representation of the unknowns to transform the semi-Lagrangian discontinuous Galerkin schemes of references [29,36] into adaptive multiresolution versions. The discontinuous Galerkin formulation enables high-order accuracy with local data for computation. It has recently been widely studied by Heath [30], Ayuso de Dios et al. [4,5], Cheng et al. [12], Rossmanith et al. [37], Restelli et al. [33], etc. Those are done with Eulerian description of the time resolution but Guo et al. [29] or Qiu and Shu [36], Crouseilles et al. [19] or Bokanowski and Simarta [10] performed semi-Lagrangian time resolution. We use multi-wavelets framework for the adaptive part (more precisely, for the multiscale representation of the distribution function of particles). Those have been heavily studied by Alpert et al. [1,2]. Some works merging multiscale

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resolution and discontinuous Galerkin methods have been described by Archibald et al. [3] for convection based problems, by Müller et al. for non-linear hyperbolic conservation laws in the finite volume framework [32,28] or for compressible flows [27]. In the framework of relativistic Vlasov equation, Besse et al. [7] presented the advantage of using adaptive meshes. While they used wavelet representation, which requires large data stencil, multi-wavelet representation coupled to discontinuous Galerkin discretisation only requires local stencil. This favours the parallelisation but an efficient parallel version of our numerical schemes (presented hereafter) will be the matter of future work. Here we start with two semi-Lagrangian discontinuous Galerkin schemes which are presented in [36] and [29]. We call them the SLDG [36] and CDG [29] schemes. We then modify these schemes to obtain adaptive multiresolution schemes with an adaptive mesh coarsening and refinement procedure. To achieve this aim, among other things detailed in Secs. 4, 5 and 6, the data are represented in a different finite-dimensional space which is constructed by using a multi-wavelet basis [1,2]. The multiscale representation, given by the multi-wavelet basis, allows to give a well-defined and natural link between local small scales of the distribution function (or data) and the local size of the adaptive mesh. Then multi-wavelet representation of data is the relevant tool to develop natural and efficient criteria for designing an adaptive mesh (refinement versus coarsening) which follows the multiscale development of the distribution function. Therefore we obtain new numerical schemes that we call AMW-SLDG (Adaptive Multi-Wavelet SLDG) and AMW-CDG.

Let us note that the SLDG and CDG schemes are equivalent when we consider the one-dimensional advection or continuity equations on a one-dimensional grid with periodic boundary conditions. The proof is given in [36], where the authors assumed that all integrals involved in the schemes are exactly computed. Nevertheless it is important to start with two non-adaptive methods which can be proven equivalent to see if our adaptive procedure introduces a bias or some differences between them. For the moment there is no mathematical proof that the AMW-SLDG and AMW-CDG schemes are equivalent. It is not only a difficult task but it remains a true open question. Actually we do not know if they are rigorously equivalent. What is certain is that the proof performed in [36] can not be extended for non-conformal adaptive n-dimensional mesh (with $n \geq 2$), which is our case. In addition in our case, where the mesh is not uniform and non-conformal, integrals are not computed exactly. Nevertheless we may expect that the AMW-SLDG and AMW-CGD schemes are close to each other, and that our procedure does not break the "symmetry" between these two schemes. A way to confirm such intuition is to make systematic comparisons of the AMW-SLDG and AMW-CDG schemes.

The paper is organised as follows. Sec. 2 presents the Vlasov–Poisson equations and their conservation properties. Sec. 3 deals with several discontinuous Galerkin methods and their application in semi-Lagrangian framework. Sec. 4 describes the multi-wavelets used to get a multiscale reconstruction of the distribution function. In Sec. 5 we identify some numerical error sources which are responsible for a loss of some conservation laws. A brief overview of numerical implementation and a summary of the full algorithm are presented in Sec. 6. Some numerical results in plasma physics and astrophysics are presented in Sec. 7. Finally, the conclusion reminds the main points of our paper and gives future perspectives of work.

2. The Vlasov–Poisson equation

We considered the following Vlasov equation

$$\begin{cases} \partial_t f(x,v,t) + \partial_x(v f(x,v,t)) + \partial_v(E(x,t) f(x,v,t)) = 0, \quad x \in \Omega_x,\ v \in \mathbb{R}, \\ f(x,v,0) = f_0(x,v), \end{cases} \tag{1}$$

where $f(x,v,t)$ is the phase-space distribution function and $E(x,t)$ is the electric field obtained from the Poisson equation

$$-\Delta\Phi(x,t) = \rho(x,t) - 1, \quad x \in \Omega_x \tag{2a}$$

$$E(x,t) = -\partial_x \Phi(x,t), \tag{2b}$$

with $\Phi(x,t)$ the electric potential and $\rho(x,t)$ the charge density linked to the distribution function f according to

$$\rho(x,t) = \int_{\mathbb{R}} f(x,v,t)\, dv. \tag{3}$$

This equation is written in adimensional variables using a fixed background of ions such that the global plasma is neutral and the integral of the distribution function on the complete phase-space is equal to 1. This system is closed using periodic boundary conditions and zero average for $E(x,t)$. The set Ω_x is a compact domain of physical space with periodic boundary conditions. It is impossible from a practical point of view to use an infinite domain in v, except if the initial distribution function is compactly supported. For this reason most of the time we consider a domain $\Omega_v = [-L, L]$ with L large enough to ensure Dirichlet boundary conditions. In our simulations, given our initial conditions, we know that the solution remains bounded in velocity space by a decreasing exponential (uniformly in x). If we take a domain large enough in the velocity direction, then integrals are small enough to ensure that the error is bounded by a small exponential. Such estimation has been used in many semi-Lagrangian codes, such as the one from [6]. We call $\Omega = \Omega_x \times \Omega_v$.