Journal of Computational Physics 332 (2017) 446–460

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

On the equivalence between the Scheduled Relaxation Jacobi method and Richardson's non-stationary method

J.E. Adsuara [a], I. Cordero-Carrión [b], P. Cerdá-Durán [a], V. Mewes [a], M.A. Aloy [a,*]

[a] Departamento de Astronomía y Astrofísica, Universidad de Valencia, E-46100, Burjassot, Spain
[b] Departamento de Matemática Aplicada, Universidad de Valencia, E-46100, Burjassot, Spain

ARTICLE INFO

Article history:
Received 12 July 2016
Received in revised form 20 October 2016
Accepted 13 December 2016
Available online 15 December 2016

Keywords:
Iterative methods for linear systems
Jacobi method
Richardson method
Scheduled relaxation Jacobi method
Finite difference methods
Elliptic equations

ABSTRACT

The Scheduled Relaxation Jacobi (SRJ) method is an extension of the classical Jacobi iterative method to solve linear systems of equations ($Au = b$) associated with elliptic problems. It inherits its robustness and accelerates its convergence rate computing a set of P relaxation factors that result from a minimization problem. In a typical SRJ scheme, the former set of factors is employed in cycles of M consecutive iterations until a prescribed tolerance is reached. We present the analytic form for the optimal set of relaxation factors for the case in which all of them are strictly different, and find that the resulting algorithm is equivalent to a non-stationary generalized Richardson's method where the matrix of the system of equations is preconditioned multiplying it by $D = \mathrm{diag}(A)$. Our method to estimate the weights has the advantage that the explicit computation of the maximum and minimum eigenvalues of the matrix A (or the corresponding iteration matrix of the underlying weighted Jacobi scheme) is replaced by the (much easier) calculation of the maximum and minimum frequencies derived from a von Neumann analysis of the continuous elliptic operator. This set of weights is also the optimal one for the general problem, resulting in the fastest convergence of all possible SRJ schemes for a given grid structure. The amplification factor of the method can be found analytically and allows for the exact estimation of the number of iterations needed to achieve a desired tolerance. We also show that with the set of weights computed for the optimal SRJ scheme for a fixed cycle size it is possible to estimate numerically the optimal value of the parameter ω in the Successive Overrelaxation (SOR) method in some cases. Finally, we demonstrate with practical examples that our method also works very well for Poisson-like problems in which a high-order discretization of the Laplacian operator is employed (e.g., a 9- or 17-points discretization). This is of interest since the former discretizations do not yield consistently ordered A matrices and, hence, the theory of Young cannot be used to predict the optimal value of the SOR parameter. Furthermore, the optimal SRJ schemes deduced here are advantageous over existing SOR implementations for high-order discretizations of the Laplacian operator in as much as they do not need to resort to multi-coloring schemes for their parallel implementation.

* Corresponding author.
E-mail addresses: jose.adsuara@uv.es (J.E. Adsuara), isabel.cordero@uv.es (I. Cordero-Carrión), pablo.cerda@uv.es (P. Cerdá-Durán), vassilios.mewes@uv.es (V. Mewes), miguel.a.aloy@uv.es (M.A. Aloy).

http://dx.doi.org/10.1016/j.jcp.2016.12.020
0021-9991/

1. Introduction

The Jacobi method [1] is an iterative algorithm to solve systems of linear equations. Due to its simplicity and its convergence properties it is a popular choice as preconditioner, in particular when solving elliptic partial differential equations. However, its slow rate of convergence, compared to other iterative methods (e.g. Gauss–Seidel, SOR, Conjugate gradient, GMRES), makes it a poor choice to solve linear systems. The scheduled relaxation Jacobi method [2], SRJ hereafter, is an extension of the classical Jacobi method, which increases the rate of convergence in the case of linear problems that arise in the finite difference discretization of elliptic equations. It consists of executing a series of weighted Jacobi steps with carefully chosen values for the weights in the sequence. The SRJ method can be expressed for a linear system $Au = b$ as

$$u^{n+1} = u^n + \omega_n D^{-1}(b - Au^n), \tag{1}$$

where D is the diagonal of the matrix A. If we consider a set of P different relaxation factors, ω_n, $n = 1, \ldots, P$, such that $\omega_n > \omega_{n+1}$ and we apply each relaxation factor q_n times, the *total amplification factor* after $M := \sum_{n=1}^{P} q_n$ iterations is

$$G_M(\kappa) = \prod_{n=1}^{P} (1 - \omega_n \kappa)^{q_n}, \tag{2}$$

which is an estimation of the reduction of the residual during one cycle (M iterations). In the former expression κ is a function of the wave-numbers obtained from a von Neumann analysis of the system of linear equations resulting from the discretization of the original elliptical problem by finite differences (for more details see [2,3]). Yang & Mittal [2] argued that, for a fixed number P of different weights, there is an optimal choice of the weights ω_n and repetition numbers q_n that minimizes the maximum *per-iteration amplification factor*, $\Gamma_M(\kappa) = |G_M(\kappa)|^{1/M}$, in the interval $\kappa \in [\kappa_{\min}, \kappa_{\max}]$ and therefore also the number of iterations needed for convergence. The boundaries of the interval in κ correspond to the minimum and the maximum weight numbers allowed by the discretization mesh and boundary conditions used to solve the elliptic problem under consideration.

In the aforementioned paper, [2] computed numerically the optimal weights for $P \leq 5$ and Adsuara et al. [3] extended the calculations up to $P = 15$. The main properties of the SRJ, obtained by [2] and confirmed by [3], are the following:

1. Within the range of P studied, increasing the number of weights P improves the rate of convergence.
2. The resulting SRJ schemes converge significantly faster than the classical Jacobi method by factors exceeding 100 in the methods presented by [2] and ~ 1000 in those presented by [3]. Increasing grid sizes, i.e. decreasing $\kappa_{\min}$, results in larger acceleration factors.
3. The optimal schemes found use each of the weights multiple times, resulting in a total number of iterations M per cycle significantly larger than P, e.g. for $P = 2$, [2] found an optimal scheme with $M = 16$ for the smallest grid size they considered ($N = 16$), while for larger grids M notably increases (e.g., $M = 1173$ for $N = 1024$).

The optimization procedure outlined by [2] has a caveat though. Even if the amplification factor were to reduce monotonically by increasing P, for sufficiently high values of P, the number of iterations per cycle M may be comparable to the total number of iterations needed to solve a particular problem for a prescribed tolerance. At this point, using a method with higher P, and thus higher M, would increase the number of iterations to converge, even if the $\Gamma(\kappa)$ is nominally smaller. With this limitation in mind we outline a procedure to obtain optimal SRJ schemes, minimizing the total number of iterations needed to reduce the residual by an amount sufficient to reach convergence or, equivalently, to minimize $|G_M(\kappa)|$. Note that the total number of iterations can be chosen to be equal to M without loss of generality, i.e. one cycle of M iterations is needed to reach convergence. To follow this procedure one should find the optimal scheme for fixed values of M, and then choose M such that the maximum value of $|G_M(\kappa)|$ is similar to the residual reduction needed to solve a particular problem. The first step, the minimization problem, is in general difficult to solve, since fixing M gives an enormous freedom in the choice of the number of weights P, which can range from 1 to M. However, the numerical results of [2] and [3], seem to suggest that in general increasing the number of weights P will always lead to better convergence rates. This leads us to conjecture that the optimal SRJ scheme, for fixed M, is the one with $P = M$, i.e. all weights are different and each weight is used once per cycle, $q_i = 1$, $(i = 1, \ldots, M)$. In terms of the total amplification factor $G_M(\kappa)$, it is quite reasonable to think that if one maximizes the number of different roots by choosing $P = M$, the resulting function is, on average, closer to zero than in methods with smaller number of roots, $P < M$, and one might therefore expect smaller maxima for the optimal set of coefficients. One of the aims of this work is to compute the optimal coefficients for this particular case and demonstrate that $P = M$ is indeed the optimal case.

Another goal of this paper is to show the performance of optimal SRJ methods compared with optimal SOR algorithms applied to a number of different discretizations of the Laplacian operator in two-dimensional (2D) and three-dimensional (3D) applications (Sect. 3). We will show that optimal SRJ methods applied to high-order discretizations of the Laplacian, which yield iteration matrices that cannot be consistently ordered, perform very similarly to optimal SOR schemes (when an optimal SOR weight can be computed). We will further discuss that the trivial parallelization of the SRJ methods outbalances the slightly better scalar performance of SOR in some cases (Sect. 3.3). Also, we will show that the optimal weight of the