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Three-dimensional shallow water system: A relaxation approach

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ABSTRACT

We study a three-dimensional shallow water system, which is obtained from the three-dimensional Navier–Stokes equations after Reynolds averaging and under the simplifying hydrostatic pressure assumption. Since the three-dimensional shallow water system is generically not hyperbolic, it cannot be numerically solved using hyperbolic shock capturing schemes. At the same time, existing simple finite-difference and finite-volume methods may fail in simulations of unsteady flows with sharp gradients, such as dam-break and flood flows. To overcome this limitation, we propose a novel numerical method, which is based on a relaxation approach utilized to "hyperbolize" the three-dimensional shallow water system. The extended relaxation system is hyperbolic and we develop a second-order semi-discrete central-upwind scheme for it. The proposed numerical method can preserve "lake at rest" steady states and positivity of water depth over irregular bottom topography. The accuracy, stability and robustness of the developed numerical method is verified on five numerical experiments.

1. Introduction

Two-dimensional (2-D) depth-averaged shallow water equations (SWEs) have been widely used in studying various hydrodynamic phenomena including dam-break flows, flood waves, and tidal flows among others. The popularity of the 2-D SWEs hinges on the following facts. First, they are substantially simpler than the three-dimensional (3-D) Navier–Stokes equations and thus can be more efficiently solved numerically. At the same time, the 2-D SWEs are capable of describing the main features of the water flow in regimes when the wave lengths are much larger than the water depth. Since the 2-D shallow water system is a hyperbolic system of balance laws, it can be accurately solved using the finite-volume methods, in particular, Godunov-type schemes, which are designed to capture both shocks and contact discontinuities. For applications of Godunov-type schemes to the 2-D SWEs, we refer the reader to, e.g., [3,4,10,16,22,25,29,34,40,50,56] and references therein.

A major drawback of the 2-D SWEs is that they are depth-averaged and thus do not take into account any vertical variation of the water flow. The 3-D shallow water system, proposed in [6,7], is still substantially simpler than the 3-D Navier–Stokes equations, and unlike the 2-D SWEs it can predict the vertical distribution of primitive variables, information

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on which is essential in many practical cases. Therefore, the 3-D SWEs have recently attracted a lot of attention as a more accurate alternative to the 2-D SWEs; see, e.g., [2,9,12,13,17,24,33,35,36,44,51].

After Reynolds averaging and under the simplifying hydrostatic pressure assumption, the 3-D SWEs with constant density and free surface are derived from the 3-D Navier–Stokes equations. The governing equations have the following form [6,7]:

$$\begin{aligned} &u_x + v_y + w_z = 0, \\ &u_t + (uu)_x + (uv)_y + (uw)_z = -g\eta_x, \\ &v_t + (vu)_x + (vv)_y + (vw)_z = -g\eta_y, \end{aligned} \tag{1}$$

where t is the time; x, y and z are the spatial coordinates; $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$ are the velocity components in the x-, y- and z-directions, respectively; g is the gravitational acceleration; and $\eta(x,y,t)$ is the water surface elevation. The system (1) together with either kinematic or dynamic boundary conditions for η constitute a closed system of equations for u, v, w and η.

In order to avoid the difficulties with applying the boundary conditions on the free surface and representing the irregular bottom, we follow the approach in [6,7,43] and replace the vertical coordinate z with the σ coordinate, which normalizes the vertical dimension to unity by the following transformation:

$$\sigma = \frac{z - \eta(x,y,t)}{D(x,y,t)},$$

in which σ is the transformed vertical coordinate that varies between -1 and 0, and $D(x,y,t)$ is the depth of the water column computed by

$$D(x,y,t) = \eta(x,y,t) - h(x,y),$$

where $h(x,y)$ is the bed level. Accordingly, the governing equations (1) can be written in the following form:

$$\eta_t + (Du)_x + (Dv)_y + \Omega_\sigma = 0, \tag{2}$$

$$(Du)_t + \left(Du^2 + \frac{g}{2}D^2\right)_x + (Dvu)_y + (\Omega u)_\sigma = -gDh_x, \tag{3}$$

$$(Dv)_t + (Dvu)_x + \left(Dv^2 + \frac{g}{2}D^2\right)_y + (\Omega v)_\sigma = -gDh_y, \tag{4}$$

where Ω represents the vertical velocity in the σ-direction, which is related to w through

$$w = \Omega + u\,(\sigma D_x + \eta_x) + v\left(\sigma D_y + \eta_y\right) + (\sigma D_t + \eta_t)\,.$$

To close the system (2)–(4), we apply the kinematic boundary conditions, namely, $\Omega(x,y,\sigma=-1,t)=0$ and $\Omega(x,y,\sigma=0,t)=0$, and integrate equation (2) with respect to σ from -1 to 0 to obtain

$$\eta_t + \left[\int_{-1}^{0} Du\,d\sigma\right]_x + \left[\int_{-1}^{0} Dv\,d\sigma\right]_y = 0. \tag{5}$$

Note that equation (2) is an elliptic equation for Ω, since using the same kinematic boundary conditions, one can rewrite equation (2) by integrating it with respect to the σ-coordinate as follows:

$$\Omega(x,y,\sigma,t) = -\int_{-1}^{\sigma}\Big[\eta_t(x,y,t) + (Du)_x(x,y,\xi,t) + (Dv)_y(x,y,\xi,t)\Big]d\xi. \tag{6}$$

The 3-D shallow water system (2)–(5) has been intensively studied and several finite difference (see, e.g., [2,11,12,17,24,33,35,36,44,51]) and finite-volume (see, e.g., [9,13]) methods for (2)–(5) have been developed. However, these methods are only applicable to smooth solutions, and thus they may fail when unsteady flows with sharp gradients such as dam-break and flood flows are simulated. This problem significantly limits the applicability of the 3-D SWEs in many problems of interest. To overcome this difficulty and extend the applicability of 3-D SWEs, one potential way is to develop shock-capturing methods for this model (this class of methods cannot be directly applied to the 3-D shallow water system, which is not hyperbolic). In order to address this issue, we propose a relaxation approach inspired by [14,47], where a hyperbolic model of compressible two-phase flow was developed using the pressure relaxation, and [1], where an unconditionally hyperbolic two-layer SWEs were obtained using the relaxation in two auxiliary layer depth variables. In §2, we present a relaxation model of the 3-D SWEs, which replaces the system (2)–(4) with the hyperbolic one that includes an additional equation on an auxiliary vertical velocity.

We then develop a second-order semi-discrete central-upwind scheme for the obtained relaxation system. Godunov-type central-upwind schemes were developed in [26,27,30,31] for general multidimensional hyperbolic systems. The central-upwind scheme was extended to the one-dimensional (1-D) SWEs in [25]. A more robust, well-balanced and at the same