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Embedded WENO: A design strategy to improve existing WENO schemes

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ABSTRACT

Embedded WENO methods utilise *all* adjacent smooth substencils to construct a desirable interpolation. Conventional WENO schemes under-use this possibility close to large gradients or discontinuities. We develop a general approach for constructing embedded versions of existing WENO schemes. Embedded methods based on the WENO schemes of Jiang and Shu [1] and on the WENO-Z scheme of Borges et al. [2] are explicitly constructed. Several possible choices are presented that result in either better spectral properties or a higher order of convergence for sufficiently smooth solutions. However, these improvements carry over to discontinuous solutions. The embedded methods are demonstrated to be indeed improvements over their standard counterparts by several numerical examples. All the embedded methods presented have no added computational effort compared to their standard counterparts.

1. Introduction

In a seminal paper in 1987, Harten and Osher introduced the essentially non-oscillatory (ENO) reconstruction technique [3]. The basic idea of ENO is to construct several different candidate polynomial interpolations and to choose the smoothest approximation to work with. The choice is facilitated by means of smoothness indicators, which become larger as the interpolation varies more rapidly.

Building on the ENO scheme, Liu, Osher and Chan introduced the weighted essentially non-oscillatory (WENO) reconstruction technique in 1994 [4]. The WENO technique comes from the realisation that the three approximations of ENO can be combined to construct a higher-order approximation. Instead of the logical statements inherent in the ENO scheme, the WENO scheme weighs every lower-order approximation according to its smoothness indicator. Thus, in smooth regions, WENO gives a better approximation, while reducing to ENO near discontinuities.

WENO schemes are ubiquitous in science and engineering, with applications in fluid dynamics, astrophysics, or any other area involving convection-dominated dynamics [5,6]. The technique is mainly applied in the context of hyperbolic and convection-dominated parabolic PDEs. However, since it is a highly advanced interpolation technique, it also has applications in fields that do not use it as part of a PDE solver, such as computer vision and image processing [7,8].

The standard WENO scheme as it is most commonly used today was devised by Jiang and Shu [1], and is sometimes referred to as the WENO-JS scheme. Recently, several variants of the WENO scheme have appeared that improve the or-

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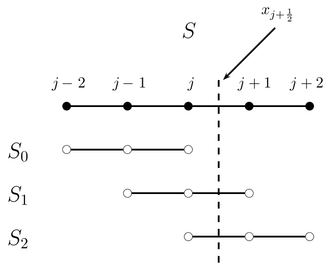

Fig. 1. The five-point stencil S, with substencils S_0, S_1 and S_2. Note that the stencil is asymmetric around the interpolation point.

der of accuracy near points where the first derivative vanishes. Examples include the WENO-M [9,10], WENO-Z [2,11,12] and WENO-NS [13] schemes. For a comparison of the performance of these schemes, see Zhao et al. [14]. Other efforts have focused on creating energy-stable WENO schemes such as those constructed by Yamaleev et al. [15,16], or decreasing numerical dissipation by using central discretisations such as considered by Hu et al. [17].

The most common implementations of WENO schemes use a five-point stencil, which can be subdivided into three three-point stencils. WENO schemes switch seamlessly between the third- and fifth-order reconstructions that are possible on the five-point stencil. The idea is straightforward: when all three smoothness indicators are roughly equal, a WENO scheme switches to the fifth-order mode. When one or more smoothness indicators are large, a WENO scheme switches to the third-order mode.

In this formulation, it seems obvious that information is discarded when only one out of three smoothness indicators is large. When this happens, the two smooth approximations could still be used to obtain better accuracy. The current WENO methods do not allow for control over the numerical solution in this situation. However, one very recent scheme which does feature this type of functionality is the targeted ENO scheme of Fu et al. [18]. Their approach is completely novel and uses a combination of ideas from ENO and WENO schemes. In our work, we propose a design strategy that aims to adapt existing WENO schemes such that they utilise the maximum number of grid points that form a smooth substencil. Moreover, we shall explicitly construct variants of two existing WENO schemes that exhibit this property.

Apart from the order of convergence, one can also analyse a WENO scheme in terms of its spectral properties [19]. WENO schemes switch non-linearly between linear modes of operation and as such, it is possible to investigate the spectral properties by analysing the underlying linear schemes [20]. We will also show that our method allows for tuning of spectral properties such as dispersion and dissipation.

This paper is arranged in the following way: in Section 2 we give a short recap of WENO methods, in Section 3 we introduce the embedding method, which is implemented in Section 4. In Section 5 we look at the spectral properties of the embedded schemes and in Section 6 we show results of several numerical experiments. Finally, we present our conclusions and outlook in Section 7.

2. The classical WENO scheme

The WENO method is an advanced interpolation technique that aims to suppress spurious oscillations. It is commonly used as part of a high-resolution scheme for hyperbolic conservation laws, e.g.,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \tag{1}$$

where f is the flux function. To obtain numerical solutions, we introduce a grid, $\{x_j\}_{j=1}^N$, with grid size Δx. With each point x_j, we associate a cell centred on x_j of width Δx, i.e., the interval $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$. Taking the average of the conservation law over cell j, we find

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + \frac{1}{\Delta x}\Big(f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t))\Big) = 0, \tag{2}$$

where u_j is the average value of u over cell j. Note that this ODE for the average value u_j is exact as long as we know the exact value of u on the cell boundaries. We shall, in the following, suppress the explicit time dependence of u, as we interpolate u in space for fixed time. In a numerical scheme, we introduce a numerical flux function to represent the fluxes on the cell edges. Regardless of the choice of numerical flux, we require the value of u at the cell interfaces $x_{j\pm\frac{1}{2}}$, i.e. $u(x_{j\pm\frac{1}{2}})$. However, if u is discontinuous and we would naively use polynomial interpolation, we inadvertently introduce spurious oscillations. A (W)ENO scheme is a more advanced interpolation technique that is designed to suppress these oscillations.

The classical WENO scheme, or WENO-JS, can be constructed by considering a five-point stencil around x_j, i.e., $S = \{x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}\}$. The large stencil can be divided into three smaller substencils, viz., $S_0 = \{x_{j-2}, x_{j-1}, x_j\}$, $S_1 = \{x_{j-1}, x_j, x_{j+1}\}$ and $S_2 = \{x_j, x_{j+1}, x_{j+2}\}$; see Fig. 1.