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Short note

Hybrid entropy stable HLL-type Riemann solvers for hyperbolic conservation laws

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ABSTRACT

It is known that HLL-type schemes are more dissipative than schemes based on characteristic decompositions. However, HLL-type methods offer greater flexibility to large systems of hyperbolic conservation laws because the eigenstructure of the flux Jacobian is not needed. We demonstrate in the present work that several HLL-type Riemann solvers are provably entropy stable. Further, we provide convex combinations of standard dissipation terms to create hybrid HLL-type methods that have less dissipation while retaining entropy stability. The decrease in dissipation is demonstrated for the ideal MHD equations with a numerical example.

1. Introduction

We consider the numerical solution of systems of hyperbolic conservation laws of the form

$$\frac{\partial \boldsymbol{q}}{\partial t} + \nabla \cdot \boldsymbol{f} = 0, \tag{1.1}$$

on a domain Ω. For a one-dimensional approximation we divide Ω into K non-overlapping grid cells $C_i = [x_{i-1/2}, x_{i+1/2}]$, $i = 1, \dots, K$ which are not necessarily equidistant. In the context of finite volume schemes, hyperbolic equations, such as (1.1), require a numerical flux function which fully determines the properties of the scheme [5]. The numerical flux function takes as input the left and right value of $\boldsymbol{q}$ at the cell interface and solves a local Riemann problem. Smooth initial flows governed by (1.1) may develop discontinuities (*e.g.* shocks) in finite time. Thus, solutions are sought in the weak sense [5]. However, weak solutions are not unique and need to be supplemented with additional admissibility criteria. Following the work of *e.g.* [10,8], we use the concept of entropy to construct discretizations that agree with the second law of thermodynamics. That is, the numerical flux function will possess entropy stability, cf. [8] and references therein.

In particular, we prove entropy stability for the HLL scheme and present the construction of HLL-type entropy stable numerical flux functions. It is known that HLL-type schemes are more dissipative than upwind schemes. However, HLL-type methods need less information about the eigendecomposition of the flux Jacobian. This is advantageous because the eigenstructure might be computationally expensive or no analytical expression exists, especially for large systems. As such, we

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consider three standard dissipation terms, namely Lax–Friedrichs (LF), HLL, and Lax–Wendroff (LW) and present two hybrid dissipation terms introduced in [7]. We demonstrate that these five schemes are provably entropy stable.

The paper is organized as follows: Sec. 2 provides a brief background on entropy stable numerical fluxes. In Sec. 3 we show entropy stability for the LF, HLL, and LW dissipation terms. The creation of two new hybrid entropy stable dissipation operators is shown in Sec. 4. We demonstrate in Sec. 5 that the new hybrid numerical flux reduce the overall dissipation in a standard finite volume scheme. Our conclusions and outlook are drawn in the final section.

2. Entropy stable numerical flux functions

A numerical method that recovers the local changes in entropy as predicted by the continuous entropy conservation law is said to be ***entropy conservative***. Entropy conservation is only valid for smooth flow configurations. For discontinuous solutions, the entropy conservation law becomes the entropy inequality [8]. A numerical scheme is said to be ***entropy stable*** as long as the numerical approximation always obeys the entropy inequality

$$\frac{\partial S}{\partial t}+\frac{\partial F}{\partial x}\leq 0, \tag{2.1}$$

where we assume that the system of hyperbolic conservation laws is equipped with a strongly convex mathematical entropy function, S, and a corresponding entropy flux, F, [8]. It is known that without additional dissipation, entropy conservative numerical schemes produce high-frequency oscillations near shocks, see *e.g.* [4,10]. Thus, for the approximation to remain valid for general flow configurations we must add a carefully designed dissipation term to ensure that (2.1) discretely holds.

To create an entropy stable (ES) numerical approximation we start with a baseline entropy conserving (EC) numerical flux and then add a dissipation term. The resulting numerical flux at an arbitrary cell interface $i+\frac{1}{2}$ takes the form

$$\boldsymbol{f}^{*,ES}=\boldsymbol{f}^{*,EC}-\frac{1}{2}\mathbf{D}\,[\![\boldsymbol{q}]\!]\,, \tag{2.2}$$

where $\boldsymbol{q}$ is the vector of conserved variables, $\mathbf{D}=\mathbf{D}(\boldsymbol{q}_i,\boldsymbol{q}_{i+1})$ is a suitable dissipation matrix evaluated at some mean state between the two cells, and $[\![\cdot]\!]=(\cdot)_{i+1}-(\cdot)_i$ is the jump between the right and left cells. For simplicity of presentation we suppress the indices on the numerical flux, the dissipation matrix, and any jump terms. For an ES scheme, the baseline central flux from a classical Riemann solver is replaced by the baseline EC flux. To guarantee entropy stability, the dissipation term in (2.2) must be carefully designed to ensure that $\boldsymbol{f}^{*,ES}$ discretely satisfies the entropy inequality (2.1). To do so, we rewrite the dissipation term [6]

$$\frac{1}{2}\mathbf{D}\,[\![\boldsymbol{q}]\!]\simeq\frac{1}{2}\mathbf{DH}\,[\![\boldsymbol{v}]\!]\,, \tag{2.3}$$

with the vector of entropy variables $\boldsymbol{v}=\frac{\partial S}{\partial \boldsymbol{q}}$ and the entropy Jacobian $\mathbf{H}=\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{v}}$ which relates the variables in conserved and entropy space. Substituting (2.3) into (2.2) the entropy stable numerical flux becomes

$$\boldsymbol{f}^{*,ES}=\boldsymbol{f}^{*,EC}-\frac{1}{2}\mathbf{DH}\,[\![\boldsymbol{v}]\!]\,. \tag{2.4}$$

The reformulation of the dissipation term, incorporating the jump in entropy variables (rather than the jump in conservative variables) makes it possible to show entropy stability [1]. From the structure of the entropy stable flux (2.4), we find a discrete version of the entropy inequality (2.1) in cell i to be [10]

$$\frac{\partial S_i}{\partial t}+\left(F_{i+\frac{1}{2}}-F_{i-\frac{1}{2}}\right)\leq-\frac{1}{2}\,[\![\boldsymbol{v}]\!]^T\,\mathbf{DH}\,[\![\boldsymbol{v}]\!]\overset{!}{\leq}0. \tag{2.5}$$

Thus, to guarantee discrete entropy stability, it is sufficient to show that $\mathbf{DH}$ is symmetric positive definite (s.p.d).

3. Entropy stable classical Riemann solvers

In this section we demonstrate entropy stability for the numerical flux of the form (2.4) for the dissipation matrix $\mathbf{D}$ of the LF, HLL, and LW scheme. To do so, we first assume that the flux Jacobian, $\mathbf{A}$, or a suitable Roe matrix, exists with the properties

$$\mathbf{A}=\mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1},\quad \mathbf{H}=(\mathbf{RZ})\,(\mathbf{RZ})^T\,, \tag{3.1}$$

where $\mathbf{R}$ is the eigenvector matrix, $\boldsymbol{\Lambda}$ the diagonal corresponding eigenvalue matrix, and $\mathbf{Z}$ is a positive diagonal scaling matrix which creates a set of entropy scaled eigenvectors $\mathbf{RZ}$ [1]. We see that, by construction in (3.1), the matrix $\mathbf{H}$ is s.p.d. In the later proofs we only use the existence of the matrices $\mathbf{R}$, $\boldsymbol{\Lambda}$, and $\mathbf{Z}$, whereas, in practice, their explicit form does not need to be known. This is advantageous, because for large systems of conservation laws, the eigendecomposition is expensive to compute or is not available.