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Finite element method for nonlinear Riesz space fractional diffusion equations on irregular domains

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ABSTRACT

In this paper, we consider two-dimensional Riesz space fractional diffusion equations with nonlinear source term on convex domains. Applying Galerkin finite element method in space and backward difference method in time, we present a fully discrete scheme to solve Riesz space fractional diffusion equations. Our breakthrough is developing an algorithm to form stiffness matrix on unstructured triangular meshes, which can help us to deal with space fractional terms on any convex domain. The stability and convergence of the scheme are also discussed. Numerical examples are given to verify accuracy and stability of our scheme.

1. Introduction

In recent years, fractional calculus is becoming more and more popular among various fields due mainly to its widely applications in science and engineering, see [1–5]. In physics, space fractional derivatives are used to model anomalous diffusion (super-diffusion and sub-diffusion). In water resources, fractional models are used to describe chemical and pollute transport in heterogeneous aquifers [6].

Owing to fractional differential equations' various applications, seeking effective methods to solve them is becoming more and more important. There is a large volume of literatures available on this subject. Researchers have presented many analytical techniques for solving fractional differential equations, such as Fourier transform method, Laplace transform method, Mellin transform method, and Green function method [5]. However, it is difficult to find the close forms of most fractional differential equations, and the close forms are always represented by special functions, such as Mittag–Leffler function, which means they are difficult to represent simply and compute directly. Moreover, most nonlinear equations are not solvable by analytical methods, so researchers have to resort to numerical methods.

Over the last few decades, many classical numerical methods have been extended to solve fractional differential equations, such as finite difference method [7–10], finite element method (FEM) [11–14], and spectral method [15–18].

As an efficient method widely used in engineering design and analysis, FEM has been deeply studied by a number of scholars to solve fractional differential equations. Ervin and Roop [13] defined directional integrals and directional derivatives, and developed a theoretical framework for the variational problem of the steady state fractional advection–dispersion equation on bounded domains in $\mathbb{R}^d$. Deng [19] investigated FEM for the one-dimensional space and time fractional Fokker–Planck equation. In [20], adopting FEM, Zhang, Liu and Anh solved one-dimensional symmetric space-

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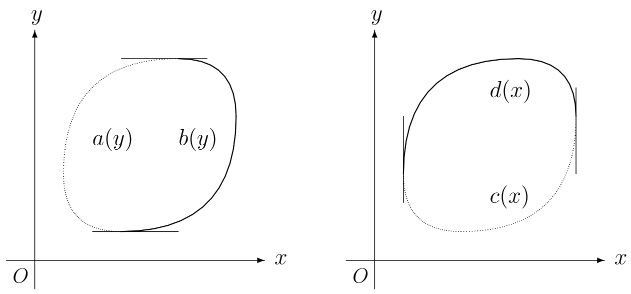

Fig. 1. Convex domain Ω with its boundary $a(y), b(y), c(x), d(x)$.

fractional differential equations. Zhang and Deng [21] proposed FEM for two-dimensional fractional diffusion equations with time fractional derivative. In [22], the authors considered FEM for the space fractional diffusion equation on domains in $\mathbb{R}$. Deng and Hesthaven [23] proposed a local discontinuous Galerkin method for the fractional diffusion equation, and offered stability analysis and error estimates. Wang and Yang [24] derived a Petrov–Galerkin weak formulation to the fractional elliptic differential equation and proved that the bilinear form is weakly coercive. Bu et al. [25–27] considered two-dimensional space fractional diffusion equations on rectangle domains solved by FEM. In [28], Qiu et al. developed nodal discontinuous Galerkin methods for fractional diffusion equations on 2D irregular domains and provided stability analysis and error estimates. Du and Wang [29] introduced a fast FEM for 2D space-fractional dispersion equations by exploiting the structure of stiffness matrix for rectangular mesh on rectangular domain. As we can see, many works on FEM are limited in solving fractional differential equations with linear source term on rectangle domains with regular meshes. Two-dimensional space fractional problems with nonlinear source term defined on irregular domains, especially partitioned with unstructured meshes, are seldom considered, although they are more real and more useful.

In this paper, we consider the two-dimensional Riesz space fractional diffusion equation on convex domain Ω with initial condition and boundary condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} = K_x \dfrac{\partial^{2\alpha} u}{\partial |x|^{2\alpha}} + K_y \dfrac{\partial^{2\beta} u}{\partial |y|^{2\beta}} + F(u) + f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T], \\ u(x,y,0) = \varphi(x,y), \quad (x,y) \in \Omega, \\ u(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times (0,T], \end{cases} \tag{1}$$

where $\frac{1}{2} < \alpha, \beta < 1$, $K_x > 0$, $K_y > 0$, and $F(u) \in C^1(\Theta)$ is a nonlinear function (Θ is a proper close domain). Boundaries of Ω are defined as follows (Fig. 1):

$$\begin{cases} a(y) = \min\{x : (x,\eta) \in \Omega, \eta = y\}, \\ b(y) = \max\{x : (x,\eta) \in \Omega, \eta = y\}, \\ c(x) = \min\{y : (\xi,y) \in \Omega, \xi = x\}, \\ d(x) = \max\{y : (\xi,y) \in \Omega, \xi = x\}. \end{cases}$$

In Eq. (1), Riesz derivatives [5] $\frac{\partial^{2\alpha} u}{\partial |x|^{2\alpha}}$ and $\frac{\partial^{2\beta} u}{\partial |y|^{2\beta}}$ are defined by

$$\begin{aligned} \frac{\partial^{2\alpha} u(x,y,t)}{\partial |x|^{2\alpha}} &= -c_\alpha \left({}_{a(y)}D_x^{2\alpha} u(x,y,t) + {}_x D_{b(y)}^{2\alpha} u(x,y,t) \right), \\ \frac{\partial^{2\beta} u(x,y,t)}{\partial |y|^{2\beta}} &= -c_\beta \left({}_{c(x)}D_y^{2\beta} u(x,y,t) + {}_y D_{d(x)}^{2\beta} u(x,y,t) \right), \end{aligned} \tag{2}$$

where $c_\alpha = \frac{1}{2\cos(\alpha\pi)}$, $c_\beta = \frac{1}{2\cos(\beta\pi)}$, and the operators ${}_{a(y)}D_x^{\mu} u(x,y)$, ${}_x D_{b(y)}^{\mu} u(x,y)$, ${}_{c(x)}D_y^{\mu} u(x,y)$, ${}_y D_{d(x)}^{\mu} u(x,y)$ ($n-1 < \mu < n, n \in \mathbb{N}$) are defined as [3]