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A symmetric Trefftz-DG formulation based on a local boundary element method for the solution of the Helmholtz equation

H. Barucq [b,a,*], A. Bendali [d,c], M. Fares [c], V. Mattesi [b,a,c], S. Tordeux [b,a,c]

[a] INRIA Bordeaux Sud-Ouest Magique-3D Team, Pau, France
[b] University of Pau, France
[c] CERFACS Algo Team, Toulouse, France
[d] University of Toulouse, Mathematical Institute of Toulouse, INSA, Toulouse, France

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ABSTRACT

A general symmetric Trefftz Discontinuous Galerkin method is built for solving the Helmholtz equation with piecewise constant coefficients. The construction of the corresponding local solutions to the Helmholtz equation is based on a boundary element method. A series of numerical experiments displays an excellent stability of the method relatively to the penalty parameters, and more importantly its outstanding ability to reduce the instabilities known as the "*pollution effect*" in the literature on numerical simulations of long-range wave propagation.

1. Introduction

When used for solving the Helmholtz equation over several hundreds of wavelengths, usual Finite Element Methods (FEM) are faced with the drawback generally called "*pollution effect*". Roughly speaking, it is necessary to augment the density of nodes to maintain a given level of accuracy, when increasing the size of the computational domain. This in turn rapidly exceeds the capacities in storage and computing even in the framework of massively parallel computer platforms (cf., for example, [1–3] and the references therein).

Several approaches have been proposed to cure this flaw. At first, for such kinds of numerical solutions, it turns out that Discontinuous Galerkin (DG) methods are more efficient than standard FEMs, also called Continuous Galerkin (CG) methods. This efficiency seems to be due in part to the less strong inter-element continuity characterizing these methods (cf., for example, [4,5]). This was confirmed in [6] where it is shown that it is possible to keep the efficiency of the DG methods by allowing discontinuities only at the interior of the elements in terms of bubble functions with penalized jumps.

Another advantage of DG methods lies in the opportunity to use shape functions, more adapted to the approximation of the solution to the interior Partial Differential Equations (PDE) of the problem, but, contrary to polynomials, with poor properties for enforcing the usual inter-element continuity conditions of the FEM. In this respect, Trefftz methods, or in

* Corresponding author at: INRIA Bordeaux Sud-Ouest Magique-3D Team, Pau, France.
E-mail addresses: helene.barucq@inria.fr (H. Barucq), abendali@insa-toulouse.fr (A. Bendali), fares@cerfacs.fr (M. Fares), vanessa.mattesi@inria.fr (V. Mattesi), sebastien.tordeux@inria.fr (S. Tordeux).

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other words methods for which the local shape functions are wave functions (cf., for example, [7,8] and the references therein), were intensively used to alleviate the aforementioned "*pollution effect*". The combination of a Trefftz method with a DG one therefore resulted in numerous approaches for solving wave equation problems called Trefftz DG method (TDG) (see, for example, [9–11,7] and the references therein).

Actually, Trefftz methods without strong inter-element continuity were used for some time now in the context of the so called Ultra Weak Variational Formulation (UWVF) devised by Després [12,13]. It was discovered later that this formulation can be recast in the context of a TDG method [14,15,9].

Some criticisms have been however addressed to the DG methods. They mainly concern the increase of the coupled degrees of freedom and a suboptimal convergence of their approximate fluxes. Hybridized versions of the DG (HDG) methods were proposed in response to these challenges [16]. However to our knowledge, HDG methods have not been used yet in the framework of a Trefftz method but only with usual local polynomial approximations [17], except in a recent paper [18], where these methods were combined in an elaborate way with geometrical optics at the element level to efficiently solve the Helmholtz equation in the high frequency regime. Since the local shape functions are only asymptotic solutions to the Helmholtz equation then, such a kind of method can be called quasi-Trefftz HDG. It is worth noting that this kind of methods cannot handle the low and the mid frequency regimes.

Instead of DG methods, some authors prefer to use a Lagrange multiplier or a least-square technique to enforce the continuity conditions (cf. [19–21]). These continuity conditions are also ensured from consistency and uniqueness arguments in the Variational Theory of Complex Ray (VTCR) method (cf. [22,7]). We have not retained these approaches in this paper, mainly because we were not convinced on the capabilities of these methods, not based on a global handling of the problem equations, to efficiently face the "*pollution effect*". However, further investigations are necessary to confirm this claim.

On the other hand, it is generally admitted that Boundary Integral Equations (BIE) lead to less "*pollution effect*" than FEMs even if to our knowledge, no formal study has confirmed such a property. Such a good behavior is probably due to the fact that BIEs can be seen as particular Trefftz methods, posed in one element, when such an interpretation is taken to the extreme. It is hence tempting to use the free space Green kernel in an approximation procedure for the interior Helmholtz equation to reduce the "*pollution effect*". This way to proceed has been already considered in [23]. However it relies upon the construction of a uniform grid in a homogeneous medium of propagation, which is manageable with very special boundaries and boundary conditions only. It seems thus difficult to extend this approach to problems involving varying coefficients or realistic geometries and boundary conditions. The aim of this study is precisely to mix two approaches by combining DG methods with BIEs, to devise a TDG method which can efficiently handle particular Helmholtz equations with varying coefficients. Specifically, either the coefficients of the Helmholtz equation are piecewise constant or they can be approximated by piecewise constant functions on a sufficiently refined decomposition of the computational domain, called *DG formulation mesh* in the rest of this paper.

The idea of building a FEM in which local shape functions are obtained on the basis of a BIE has been recently investigated in [24,25]. Our work stands out from these approaches in using DG framework and an improved approximation of the Dirichlet-to-Neumann (DtN) operator. The latter is a key feature here for matching the local solutions at the interfaces of the mesh. The corresponding method can be viewed globally as a DG method at the level of the DG formulation mesh and locally as a BIE at the element level. Actually, BIEs are used only to compute the DtN operator within each element of the DG formulation mesh. As shown below, the quality of the overall solution strongly depends on the accuracy of the approximation of the DtN operator. Specific numerical procedures have therefore been developed to increase the accuracy of this approximation. Similar techniques were considered in [26,27]. Our method is also formulated as a symmetric variational formulation of the corresponding boundary-value problem. Its derivation is inspired by [28] (see also [29, p. 122]) where Symmetric Interior Penalty (SIP) methods have been designed. The symmetry yields an important gain. The storage of the boundary integral operators involved in the formulation is indeed avoided, the contribution of the BIEs being element-wise only. It is also worth noting that all the degrees of freedom of the discrete problem to be solved are located on the "skeleton" of the mesh, that consists of the boundaries of the elements of this mesh. Such a feature moves towards HDG methods even if it keeps with unknowns on both sides of the interfaces. Note as well that our choice of the local approximating functions allows the approximation of both the propagating and evanescent waves in a natural way. It should be noted also that, even if the method, considered here, is of Trefftz type, the local approximations are done by means of a Boundary Element Method (BEM) (cf., for example, [30,31]). As a result, these approximations are ultimately performed in terms of piecewise polynomial functions on a BEM mesh. In contrast then to usual Trefftz methods (cf., for example, [19,11,32,9,13] to cite a few), h or p refinements are as simple and efficient as in a standard FEM. In the following, this method is called the BEM Symmetric Trefftz DG method and more concisely denoted by BEM-STDG.

The paper is organized as follows. In Section 2, after stating the boundary-value problem, we first derive the variational formulation of the symmetric TDG method and show how it can be connected to previous DG formulations. Section 3 develops the BEM procedure used to define the Trefftz method. Section 4 is devoted to the numerical validation of the method in two dimensions, while Section 5 deals with the comparison of its performances with a standard Interior Penalty DG (IPDG) method based on element-wise polynomial approximations. A final brief section gives some concluding remarks and indicates further studies that can extend the current one.

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