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Fractional Burgers equation with nonlinear non-locality: Spectral vanishing viscosity and local discontinuous Galerkin methods

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ABSTRACT

We consider the viscous Burgers equation with a fractional nonlinear term as a model involving non-local nonlinearities in conservation laws, which, surprisingly, has an analytical solution obtained by a fractional extension of the Hopf–Cole transformation. We use this model and its inviscid limit to develop stable spectral and discontinuous Galerkin spectral element methods by employing the concept of spectral vanishing viscosity (SVV). For the global spectral method, SVV is very effective and the computational cost is $O(N^2)$, which is essentially the same as for the standard Burgers equation. We also develop a local discontinuous Galerkin (LDG) spectral element method to improve the accuracy around discontinuities, and we again stabilize the LDG method with the SVV operator. Finally, we solve numerically the inviscid fractional Burgers equation both with the spectral and the spectral element LDG methods. We study systematically the stability and convergence of both methods and determine the effectiveness of each method for different parameters.

1. Introduction

The classical integer-order Burgers equation is a fundamental partial differential equation employed for modeling interesting dynamics in fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow. Recently, considerable interest in fractional Burgers equations has been stimulated due to their application in the areas of over-driven detonation in gas [1], anomalous diffusion in semiconductor growth [2], hereditary effects on nonlinear acoustic waves [3], nonlinear Markov processes propagation of chaos [4], etc. Some more interesting studies can be found in [5–11] and references therein.

In the majority of these studies of the viscous fractional Burgers equation there are two extensions of the classical form. In the first one, the standard diffusion term is replaced by the fractional derivative operator, in the form

$$u_t + \frac{1}{2}(u^2)_x = \varepsilon D^{\alpha} u,\ 1 < \alpha < 2, \tag{1.1}$$

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where D^α denotes the fractional derivative operator, e.g., the Riesz fractional operator [10], or the fractional non-local operator defined by the Fourier transform [5–8]. Another generalization is introduced by modifying the first-order time derivative u_t with a fractional derivative, i.e., ${}_0D_t^\nu u$, which gives [12]

$$ {}_0D_t^\nu u + \frac{1}{2}(u^2)_x = \varepsilon u_{xx},\ 0 < \nu < 1. \tag{1.2}$$

In both fractional Burgers equations (1.1) and (1.2), the non-local fractional operators are associated with linear terms. Non-local conservation laws whose non-locality associated with nonlinear term have been studied in [13] using an extension of the peridynamic theory of continuum mechanics. Numerical results of the nonlocal Burgers equation discussed in [13] showed that the choice of kernel function can regularize the solution of the nonlocal inviscid Burgers equation such that a shock will not develop. In [13], the nonlinear non-locality is characterized by spatially integral operators that do not involve derivatives and the viscous Burgers equation can be viewed as a limit of the non-local Burgers equation in an appropriate way. Recently, by adapting the concept of fractional derivatives, Miškinis in [14], based on a fractional generalization of the Hopf–Cole transformation, presented another type of fractional Burgers equation (FBENN) by using Caputo fractional derivatives, which is nonlinear and represents an interesting non-local generalization of the Burgers equation, that is

$$ u_t + \frac{1}{2}{}_aD_x^{\beta_1}({}_aD_x^{\beta_2}u)^2 = \varepsilon u_{xx},\ 0 \le \beta_1, \beta_2 \le 1,\ \beta_1 + \beta_2 = 1, \tag{1.3}$$

where the fractional derivatives are left fractional derivatives in the Caputo sense. If $\beta_1 = 1$, $\beta_2 = 0$, then (1.3) degenerates to the classical Burgers equation. Miškinis studied the exact solutions, the conservation laws, associated symmetries, and the asymptotic form of solutions to problem (1.3). One of the most interesting theoretical results is that the exact analytic solution of problem (1.3), which is

$$ u(x,t) = -2\varepsilon\, {}_aD_x^{\beta_1}\Big[\log\Big(1 + \frac{1}{\sqrt{4\pi\varepsilon t}}\int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4\varepsilon t} - \frac{1}{2\varepsilon}{}_aI_y^{\beta_1}u_0(y)}dy\Big)\Big] \tag{1.4}$$

can be obtained by the fractional generalized Hopf–Cole transformation [14]

$$ u(x,t) = -2\varepsilon\, {}_aD_x^{\beta_1}\log(1 + w(x,t)), \tag{1.5}$$

where $w(x,t)$ is the solution of the diffusion equation $w_t = w_{xx}$ with initial condition $w(x,0) := w_0(x) = e^{-\frac{1}{2\varepsilon}{}_aI_x^{\beta_1}u_0(x)}$. The classical conservation laws describe behavior that is governed by point-values of the state, its derivatives and the flux. On the other hand, there exist physical theories in which values of some quantity at a point are influenced by values of the field in a neighborhood of that point. Such theories are generically referred to as "nonlocal" [13]. The model (1.3) also enjoys such kind of property of a conservation law in the sense that the fractional derivative of the solution is conserved. For instance, if $\forall t > 0$, ${}_aD_x^{2-\beta_1}u(\pm\infty, t) = {}_aD_x^{\beta_2}u(\pm\infty, t) = 0$, define

$$ I^{(\beta_1)} = \int_{-\infty}^{\infty} {}_aD_x^{1-\beta_1}u(x,t)dx, $$

then we have the following conservation law:

$$ \frac{\partial I^{(\beta_1)}}{\partial t} = 0. \tag{1.6}$$

Hence, the model of (1.3) provides an excellent testbed to develop and test new numerical methods for nonlinear conservation laws with a non-local nonlinear flux introduced through fractional differential operators – a topic that has not been addressed so far in the literature of fractional PDEs. To this end, we are interested to study how the solutions of fractional nonlinear conservation laws, which may develop spontaneous jump discontinuities, i.e., shock waves, vary with the fractional order for the viscous model but also for its inviscid fractional limit. In particular, we are interested in developing high-order spectral approximations to (1.3). Due to the presence of discontinuities, the spectral approximations may experience spurious Gibbs oscillations, which, in turn, may lead to loss of high accuracy and most importantly loss of stability. In numerical implementations for standard conservation laws, spectral methods are often augmented with smoothing procedures in order to reduce the Gibbs oscillations [15] associated with discontinuities arising at the boundaries or due to under-resolution. However, for nonlinear problems, convergence of spectral approximations may fail despite the additional smoothing of the solution.

The aim of this paper is to introduce the concept of spectral vanishing viscosity (SVV) into spectral and spectral element LDG approximations for problem (1.3) to obtain stable and highly accurate solutions. We design an efficient SVV approach with the nonlinear non-local term discretized properly using ideas from fractional calculus and Jacobi polynomials. Moreover, by rewriting equation (1.3) into a suitable system, we present a LDG approximation to (1.3). The rest of this paper is