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Higher-order triangular spectral element method with optimized cubature points for seismic wavefield modeling

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ABSTRACT

The mass-lumped method avoids the cost of inverting the mass matrix and simultaneously maintains spatial accuracy by adopting additional interior integration points, known as *cubature* points. To date, such points are only known analytically in tensor domains, such as quadrilateral or hexahedral elements. Thus, the diagonal-mass-matrix spectral element method (SEM) in non-tensor domains always relies on numerically computed interpolation points or quadrature points. However, only the *cubature* points for degrees 1 to 6 are known, which is the reason that we have developed a *p-norm*-based optimization algorithm to obtain higher-order cubature points. In this way, we obtain and tabulate new cubature points with all positive integration weights for degrees 7 to 9. The dispersion analysis illustrates that the dispersion relation determined from the new optimized cubature points is comparable to that of the mass and stiffness matrices obtained by exact integration. Simultaneously, the Lebesgue constant for the new optimized cubature points indicates its surprisingly good interpolation properties. As a result, such points provide both good interpolation properties and integration accuracy. The Courant–Friedrichs–Lewy (CFL) numbers are tabulated for the conventional Fekete-based triangular spectral element (TSEM), the TSEM with exact integration, and the optimized cubature-based TSEM (OTSEM). A complementary study demonstrates the spectral convergence of the OTSEM. A numerical example conducted on a half-space model demonstrates that the OTSEM improves the accuracy by approximately one order of magnitude compared to the conventional Fekete-based TSEM. In particular, the accuracy of the 7th-order OTSEM is even higher than that of the 14th-order Fekete-based TSEM. Furthermore, the OTSEM produces a result that can compete in accuracy with the quadrilateral SEM (QSEM). The high accuracy of the OTSEM is also tested with a non-flat topography model. In terms of computational efficiency, the OTSEM is more efficient than the Fekete-based TSEM, although it is slightly costlier than the QSEM when a comparable numerical accuracy is required.

1. Introduction

The spectral element method (SEM) is a spectrally accurate algorithm for solving partial differential equations (PDEs), which combines the geometrical flexibility of the finite element method (FEM) with the accuracy associated with the spec-

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tral method [1]. The computational domain can be divided into quadrilateral (hexahedral) [2–4] or triangular (tetrahedral) elements [5–8]. All variables in each element are approximated by high-order polynomials. The solution of discrete PDEs is obtained by using their integral forms. The diagonal mass matrix can be obtained with appropriate discretization elements as well as clever choice of test functions and collocation points (i.e., collocated interpolation and integration points), which leads to a computationally efficient method [2].

The quadrilateral spectral element method (QSEM) relies on a nodal basis derived from Gauss–Lobatto–Legendre (GLL) points (or the tensor products of GLL points) [2]. The GLL points have both near-optimal polynomial interpolation and integration (cubature) properties [9], which allow one to integrate the stiffness matrix of degree $2N-2$ exactly and to estimate the mass matrix of degree $2N$ using a high-precision quadrature rule with algebraic accuracy of $2N-1$ [9,10]. Simultaneously, the small Lebesgue constant of the GLL points means that they can generate a well-conditioned nodal basis. Although a quadrilateral grid has been successfully used, it cannot faithfully and flexibly represent extremely complex geometry. Compared to the quadrilateral and hexahedral elements, the triangular and tetrahedral elements are considerably more flexible for practical applications. Despite it is possible to divide each triangle (or tetrahedron) into four quadrilaterals (or hexahedra) at the expense of increasing the number of nodes per element, the quality of the resulting meshes tends to be poor [11]. Therefore, it seems advisable to develop a method that can be directly applied to the simplex.

Bearing highly complex geometries in mind, the SEM on triangular (tetrahedral in 3D) elements is generally preferred [5–8,12–17]. Unfortunately, points analogous to the GLL points in the case of triangle have not been found after 120 years of research on orthogonal polynomials [18]. At present, the points with an optimal polynomial interpolation and integration properties are only known analytically for an interval and their tensor products. Hence, SEM is usually confined to quadrilateral and hexahedral elements. For non-tensor product domains, such as triangular or tetrahedral elements, there is very little analytical knowledge about the location of the optimal points. This problem is still an open issue. As the GLL points in a one-dimensional (1D) case cannot be extended to a two-dimensional (2D) case, the diagonal mass matrix methods on the triangle always resort to numerically computed interpolation points [13,19–21], or quadrature points [14]. Until now, the typical methods used to find these points either optimize the interpolation nodes or the polynomial integration rather than both. In this sense, several attempts have been made to determine these points by minimizing the Lebesgue constant directly or indirectly. Chen and Babuška [20] directly minimized the Lebesgue constant. Bos [19] and Taylor et al. [13] chose node positions that maximize the determinant of the classical Vandermonde matrix, and the resulting nodes are known as *Fekete* points. An alternative and physically motivated approach comes from the observation of Stieltjes [22,23], which indicates that the roots of the Jacobi polynomials coincide with the equilibrium configuration of charges constrained to lie on the bi-unit interval. Hesthaven [21] extended this analogy to compute nodal distributions by seeking equilibrium positions of charges distributed on the triangle with line charges on the boundaries. Some explicit approaches have also been proposed using an easy-to-implement scheme. Warburton [24] made an explicit construction for interpolation nodes on a simplex whose Lebesgue constants are better than or comparable to those of the existing node sets, at least up to the tenth-order interpolation. Blyth et al. [25,26] obtained a Lobatto interpolation grid over the triangle by means of a sequence of increasingly refined grids, whose Lebesgue constants compete with those of more complicated nodal distributions. Following Blyth's [25] idea, Luo and Pozrikidis [27] constructed a Lobatto interpolation grid on a tetrahedron. Pasquetti and Rapetti [17] reviewed the choice of interpolation nodes on the triangle.

For the optimal interpolation points on the triangle, such as the Fekete points [12,19], electrostatic points [21] and minimum Lebesgue constant points [20], although they are all well-conditioned interpolation nodes, the algebraic accuracy of the generalized Newton–Cotes integration based on these nodes reaches only degree N (i.e., the order of the Lagrange polynomials on the triangle) [9,28]. This is a poor approximation of the inner products of the stiffness matrix (degree $2N-2$) and the mass matrix (degree $2N$) for seismic wave modeling or second-order PDEs [9,10]. Therefore, the poor approximation to the stiffness and mass matrices using the integration formula with accuracy of degree N (i.e., the generalized Newton–Cotes integration) is insufficient to achieve an exponential convergence [29] because the optimal interpolation points are only good for polynomial interpolation but not for the integration property.

To date, some attempts have been made to find a quadrature rule that can integrate a larger space. Cohen et al. [30] made a pioneering work by enriching the polynomial space with additional interior nodes that vanish at the edges and vertices of the elements. They constructed the points that are analogous to the GLL points for degrees 2 and 3 on the triangle, which increases the integration precision and has positive integration weights. Following the similar idea, Mulder [31–33] obtained such points for degrees 4 to 6, and Giraldo and Taylor [9] for degrees 6 to 7. This method is the so-called mass-lumped method, and the integration points are called *cubature* points because they have higher integration accuracy in contrast to the optimal interpolation points. Although Giraldo and Taylor [9] gave the cubature points for degrees 6 to 7, these points are highly unstable for seismic wavefield modeling. For this reason, they adopted a strong low-pass filter (i.e., erfc-log filter) to stabilize the solution at every 20 time steps. In addition, Mulder [33] pointed out that the integration strength given by Giraldo and Taylor [9] is too small. Furthermore, Helenbrook [34] theoretically proved the following property of the integration rule on the triangle: an integration rule with $(N+1)(N+2)/2$ integration nodes is not capable of exactly integrating the space spanned by $T(2N-1) \equiv \{x^m y^n | 0 \le m, n; m+n \le 2N-1\}$, where N is the order of interpolant polynomials; the integration nodes must include three vertices, $N+1$ points on each edge, and $(N-2)(N-1)/2$ points in the interior of the triangle. Following the Helenbrook's work [34], Xu [35] proved that the number of inner-mode nodes must be greater than or equal to $N(N-1)/2$ if a cubature rule with algebraic accuracy of $2N-1$ exists on the triangle.