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Chebyshev-like generalized Shapiro filters for high-accuracy flow computations

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ABSTRACT

This paper presents Chebyshev-like generalized Shapiro (CS) filters with improved spectral-like resolution compared to existing generalized Shapiro filters. These new filters combine the advantages of Shapiro filters, *i.e.* arbitrary accuracy order, no-dispersion, full damping of 2Δ-waves, and the advantages of Chebyshev filters, *i.e.* purely dissipative response function with equal ripples satisfying an arbitrary Chebyshev criterion in passband. Thanks to the formalism of generalized Shapiro filters, general formulas are derived for arbitrary accuracy orders and arbitrary Chebyshev criterion. A python script is provided in appendix to compute CS filter coefficients. Computations based on the Euler equations assess the benefit of CS filters compared to the standard Shapiro filters. Since CS filters differ from Shapiro filters only by their coefficients, they can easily and advantageously be implemented in computational solvers already making use of generalized Shapiro filters.

1. Introduction

Spatial filters are used in many computational physic problems and are mostly designed for two distinct applications: either to smooth an initial data field or to ensure stability and convergence of computations performed with non-dissipative finite-volume or finite-difference high-order schemes. In both cases, their role is to cancel spurious waves, or 2Δ-waves, while preserving long-wavelength components of the filtered fields. Since the late 1950's and early work of Shuman [1], there has been a constant research effort to design more accurate and selective filters, mostly in the fields of weather forecasting and climate modeling [2], computational fluid dynamics [3] and computational aeroacoustics [4]. Following the ideas of Shuman, Shapiro [5–7] designed in the 1970's a class of optimal explicit filters of $2n$ order accuracy, defined on $(2n+1)$-point stencil, that preserves the largest structures and fully damps the 2Δ-waves. Since the breakthrough of Shapiro filters, two paths have been employed to increase the cut-off frequency (or passband) of selective filters. The first path, already used by Shuman [1], consists in giving up the maximal accuracy achievable by the filter on a given stencil and accepting a "small" amount of error in passband in trade off an increase of the cut-off frequency. This path has been followed by von Storch [8] and more recently by Bogey and Bailly [9], Bogey, de Cacqueray and Bailly [10] or Redonnet and Cunha [11]. The second path relies on the use of Padé fractions that allow more flexibility on the filter design at the cost of linear algebra. Such filters have been designed by Raymond [12–14], Lele [15], Gaitonde, Visbal *et al.* [16,17], and Kim [18]. These filters are originally designed to be used on structured grids but recent studies show that it is possible to extend them to unstructured grids [19]. More details on filter design, advantages, or limitations compared to other methodologies

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for weather forecasting, climate modeling, computational fluid dynamics and computational aeroacoustics can be found in Refs. [2–4,14,20,21].

The work presented here follows the first of the two paths mentioned above and relies on a analysis of the design criteria, strengths, and weaknesses of existing spectral-like filters. Based on this analysis and taking advantage of the formalism of generalized Shapiro filters introduced in Ref. [22], we derive a new class of spectral-like filters that combines the advantages of Shapiro filters [5–7], *i.e.* arbitrary accuracy, no-dispersion, full damping of 2Δ-waves, and the advantages of Chebyshev filters of the first type [20,21], *i.e.* purely dissipative response function with equal ripples satisfying an arbitrary Chebyshev criterion in passband. These filters will be referred to as Chebyshev-like generalized Shapiro filters or CS filters. The paper is organized as follows. The Shapiro filters are recalled in section 2. Section 3 summarizes the formalism of generalized Shapiro filters. In section 4, an analysis of the design criteria and properties of existing spectral-like filters is performed. Section 5 describes the design principles and properties of the Chebyshev-like generalized Shapiro filters. Generic formulas are derived for arbitrary accuracy orders and arbitrary Chebyshev criterion. A python script, computing CS filter coefficients, is provided in Appendix A. In section 6, the benefit of CS filters compared to standard Shapiro filters is assessed through the computations of steady and unsteady fluid dynamics test cases: the numerical evolution of a stationary vortex, the advection of a vortex, and a smooth bubble convection.

2. Shapiro filters

Let $w = w(x)$ be a continuous function defined for $x \in \mathbb{R}$ and, $w_i = w(x_i)$ be its discretization on a uniform grid ($x_i = i\Delta x$) with spatial step Δx. Applying a $(2n+1)$-point nonrecursive symmetric filter F to w provides:

$$\tilde{w}_i = F(w_i) = \sum_{k=-n}^{n} a_k w_{i+k} \tag{1}$$

where $\tilde{w}_i$ is the filtered value of w at point $x_i = i\Delta x$, and a_k are the filter coefficients. Shapiro filters [5–7] are $2n$-order accurate $(2n+1)$-point filters of the form:

$$S^{2n}(w_i) = \left(1 + (-1)^{n-1}(\delta/2)^{2n}\right) w_i \tag{2}$$

where n is a positive integer, δ is the standard finite difference operator [23,24], defined by $\delta w_i = w_{i+1/2} - w_{i-1/2}$, and δ^{2n} denotes $2n$ successive applications of δ. The discrete coefficients of the operator $\delta^{2n}/2^{2n}$ are given by the $(2n+1)$-row entries of Pascal's triangle corresponding to the development of $(a-b)^n$ divided by 2^{2n}. The response function of the filter, $\widehat{S}^{2n}(\xi)$, and its Taylor expansion writes respectively:

$$\widehat{S}^{2n}(\xi) = 1 - \sin^{2n}(\xi/2) \tag{3}$$

with $\xi = k\Delta x$ the reduced wavenumber, and

$$S^{2n}(w_i) = w_i + (-1)^{n-1}\frac{\Delta x^{2n}}{2^{2n}}\left.\frac{\partial^{2n} w}{\partial x^{2n}}\right|_i + \mathcal{O}\left(\Delta x^{2n+2}\right) \tag{4}$$

Summarizing the analysis given by Shapiro in Ref. [6], these one-dimensional filters are equivalent to the identity operator plus a linear diffusion of order $2n$ with a coefficient $K_n = (-1)^{n-1}\Delta x^{2n}/2^{2n}$. Often referred to as standard explicit filters, Shapiro filters are the corner stone of a wide class of explicit and implicit generalized Shapiro filters [22] used as substitute for numerical dissipation in weather forecasting or climate modeling [2], high-order computational fluid dynamics [3] and computational aeroacoustics [4].

3. Generalized Shapiro filters

Initially restricted to filters discussed by Purser [25], the notion of generalized Shapiro filters has been recently extended to other explicit and implicit filters in Ref. [22] using a formalism that makes easier their analysis and design. Restricting the forthcoming reminder to explicit filters, a linear filter F defined on a $(2n+1)$-point stencil is a generalized Shapiro filter if it can be expressed as:

$$F(w_i) = \left(1 + \sum_{k=1}^{n} A_{2k} D^{2k}\right) w_i \quad \text{with} \quad D^{2k} = (-1)^{k-1}\frac{\delta^{2k}}{2^{2k}} \quad \text{and} \quad \sum_{k=1}^{n} A_{2k} = 1 \tag{5}$$

where D^{2k} is the dissipative operator of the $2k$-th order Shapiro filter and where A_{2k} are real coefficients. An explicit filter is thus a generalized Shapiro filter if it is a linear combination of Shapiro filters with the sum of the A_{2k} coefficients being equal to one (which ensures the cancellation of the 2Δ-waves). Thus for explicit filters, an equivalent notation is: