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# A blended continuous–discontinuous finite element method for solving the multi-fluid plasma model

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ABSTRACT

The multi-fluid plasma model represents electrons, multiple ion species, and multiple neutral species as separate fluids that interact through short-range collisions and long-range electromagnetic fields. The model spans a large range of temporal and spatial scales, which renders the model stiff and presents numerical challenges. To address the large range of timescales, a blended continuous and discontinuous Galerkin method is proposed, where the massive ion and neutral species are modeled using an explicit discontinuous Galerkin method while the electrons and electromagnetic fields are modeled using an implicit continuous Galerkin method. This approach is able to capture large-gradient ion and neutral physics like shock formation, while resolving high-frequency electron dynamics in a computationally efficient manner. The details of the Blended Finite Element Method (BFEM) are presented. The numerical method is benchmarked for accuracy and tested using two-fluid one-dimensional soliton problem and electromagnetic shock problem. The results are compared to conventional finite volume and finite element methods, and demonstrate that the BFEM is particularly effective in resolving physics in stiff problems involving realistic physical parameters, including realistic electron mass and speed of light. The benefit is illustrated by computing a three-fluid plasma application that demonstrates species separation in multi-component plasmas.



## 1. Introduction

Plasmas can be represented by a hierarchy of models; the more general the model, the higher the computational cost. In plasma simulations it is therefore important to devise methods that maximize computational efficiency, while capturing the desired physics.

In kinetic theory, each constituent plasma species is represented by a probability distribution function $f(\mathbf{x}, \mathbf{v}, t)$ that depends on position, velocity, and time. The evolution of the distribution function is governed by the Boltzmann–Maxwell equation system. Solving the Boltzmann equation is computationally expensive due to the fact that the distribution functions occupy a six-dimensional phase space.

The two-fluid plasma model can be derived from the kinetic model by taking velocity moments [1], which reduces the six-dimensional space to three dimensions. Inherent in the derivation of the two-fluid model is the assumption of local thermodynamic equilibrium within each species, but not between different species. The governing equations for the two-fluid model are derived by taking the first three velocity moments of the Boltzmann equation for the electrons and for

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the ions. The zeroth moment describes the conservation of mass, the first moment describes the conservation of momentum, and the third moment describes the conservation of energy. The moments of the Boltzmann equation describe the evolution of the bulk properties of the plasma: density, momentum, and energy. In the simplest two-fluid description, the pressure is assumed to be isotropic and the heat flux is assumed to be negligible [1].

The magnetohydrodynamics (MHD) model, the most widely used plasma model, is derived from the two-fluid plasma model by neglecting the electron inertia (zero electron mass) and assuming the speed of light is much larger than any other speed in the system (infinite speed of light) [2]. As a consequence of neglecting the electron inertia, the electron momentum equation reduces to the generalized Ohm's law and the kinetic energy of the electrons is zero. By making these asymptotic assumptions, the physics of high frequency electromagnetic waves is ignored and the vacuum permittivity is effectively set to zero. This means that the displacement current term of Ampere's law is zero, and from Poisson's equation a zero permittivity implies that the electron and ion number density must always be equal, thereby enforcing charge neutrality. The MHD model is often further simplified to an ideal MHD model, which limits its applicability to high collisionality, small Larmor radius, and low resistivity regimes [3].

The two-fluid plasma model can be generalized to a multi-fluid plasma model that includes multiple ions and neutral species, where the mass, momentum, and energy of each species is evolved separately, and the species interact with each other through collisions and electromagnetic fields [4,5]. By separately evolving the constituent species, the multi-fluid plasma model is able to capture more generalized physics than MHD, but at a higher computational cost. The mass of the constituent species and their plasma parameters set the range of spatial and temporal scales. Since the multi-fluid plasma model does not make asymptotic assumptions about the speed of light, it captures more waves than MHD, including waves that propagate faster than the magnetosonic speed. This has been demonstrated for the two-fluid plasma model [1].

The characteristic speeds of the multi-fluid plasma model are the species' acoustic speeds and the speed of light, both of which can severely limit the time step size for the numerical time integration. In addition, the characteristic frequencies (plasma and cyclotron) need to be resolved to capture the full physics of the multi-fluid model.

For a given model, the partial differential equation (PDE) type informs the choice of numerical methods used to solve it. The multi-fluid plasma model is an inhomogeneous hyperbolic equation system and can be described by balance laws. Such equation systems can be solved using a variety of methods, including finite volume methods [6–8], continuous Galerkin finite element methods [9,10], and discontinuous Galerkin finite element methods [11–14].

Finite volume methods have been used extensively and differ depending on the technique used to evaluate fluxes. One type of finite volume method is the wave propagation method, which is second-order accurate and provides good resolution of shocks and discontinuities even when the initial conditions are smooth [6]. Other types of finite volume methods have been successfully applied to the MHD plasma model [15–17] and to the two-fluid plasma model [1,18]. Since the source terms in the PDEs cannot be directly incorporated into the calculation of the fluxes in the wave propagation method, the approach requires source splitting. As a result phase errors can be produced when the characteristic frequency is high compared to the frequency of information propagation [14].

Continuous Galerkin (CG) finite element methods have also been used for solving MHD [9] and extended MHD equations [10]. The CG method represents the solution variables within each element using polynomial basis functions. The order of the polynomial determines the spatial order of accuracy. The CG method enforces continuity of the solution across element boundaries, i.e. $C^0$ continuity. Some CG methods enforce $C^1$ continuity of the solution across element boundaries [9]. The CG method is particularly well suited for smooth solutions and offers the ability to compute solutions at high-order spatial accuracy on regular and unstructured grids [19–22]. With no dissipation, CG methods can be prone to dispersive errors and often require adding an artificial dissipation to damp high frequency oscillations [23]. The ideal multi-fluid plasma model has no physical source of dissipation, and thus using a CG method necessitates the introduction of artificial dissipation.

CG methods require the simultaneous solution of the global system of equations, which involves a matrix inversion. This feature allows CG to be coupled with an implicit time integration method with only minor modification. Plasma dynamics encompass a large range of timescales, which makes implicit time integration desirable. With implicit time integration, the solver is not subject to CFL (Courant–Friedrichs–Lewy) restrictions that limit time step size based on the fastest speed in the system. As two relevant plasma examples, in Ref. [24] an implicit method is applied together with a CG spatial discretization, and in Ref. [25] the hyperbolic MHD model is converted into parabolic equations in order to make them more amenable to multigrid and physics-based preconditioning that allow for fast Jacobian-free implicit time integration.

A numerical method that combines the shock capturing and conservation properties of finite volume methods with high-order accuracy and flux/source coupling of CG methods is the discontinuous Galerkin (DG) finite element method. The DG method was introduced in Ref. [26] for the study of two-dimensional neutron transport. Like the CG methods, DG methods represent the solution by a set of polynomial basis functions in each element; however, continuity of the solution is not enforced across the element boundaries. The DG method was expanded to solve non-linear equations in Ref. [11] who used it with total variation diminishing (TVD) Runge–Kutta time integration. Likewise the DG method has been applied to solve Navier–Stokes equations [27] on unstructured grids with linear, quadratic, and cubic elements [28–30].

In plasma simulations the DG method has been applied to the ideal MHD model [12,13,31], and to the Vlasov–Poisson equation system [32]. In Ref. [14] an extensive study of the DG method is applied to the two-fluid plasma model and explores the challenges associated with capturing the physical dispersion of the model. It is shown that the DG method is able to accurately capture physically expected high frequency oscillations using higher order discretizations without producing phase errors. A major benefit of the DG method is that it is remarkably robust in the presence of rapid oscillations while