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Short note

Canonical straight field line magnetic flux coordinates for tokamaks

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ABSTRACT

New global straight field line coordinates are introduced for a toroidal plasma configuration. The new coordinate system provides a canonical description of particle guiding center motion while maintaining the straight field line feature. These coordinates are convenient for combining MHD calculations with kinetic modeling of energetic particles. We demonstrate how the new coordinate system can be constructed by transforming the poloidal and toroidal angles. Numerical examples show comparison of the new coordinates with various non-canonical coordinates for the same equilibrium configuration.

1. Introduction

Tokamaks are designed to have closed, nested magnetic surfaces on which plasma pressure is constant as required by the force balance. The Grad–Shafranov equation determines configuration of these surfaces for the tokamak equilibrium [1]. Due to strong anisotropy associated with the magnetic field, the magnetic surface label ψ is a convenient radial coordinate for computations. The corresponding commonly used magnetic flux coordinates (ψ, θ, ζ) are the magnetic flux, the generalized poloidal and toroidal angles. In most cases, the generalized angles (θ, ζ) do not have to be the geometric poloidal and toroidal angles (θ_g, ϕ). For convenience, the generalized angles are often chosen so that the magnetic field lines are straight when plotted on (θ, ζ) plane, i.e.:

$$\mathbf{B} = \chi'(\nabla\zeta \times \nabla\psi + q(\psi)\nabla\psi \times \nabla\theta),$$

where $q(\psi)$ is the safety factor, $\chi(\psi)$ is the poloidal magnetic flux with prime denoting the derivative with respect to ψ, and $\mathbf{B} \cdot \nabla\psi = 0$. These coordinates are the so-called straight field line coordinates and they are particularly convenient for stability analysis. The coordinate system in which the field lines are straight is not unique. The commonly used versions of the straight field line coordinates are PEST [2], Hamada [3], equal arclength, and Boozer [4] coordinates.

It should, however, be noted that these four versions are not optimal for Hamiltonian description of the particle guiding center motion in tokamaks. A Hamiltonian description of the guiding center equations of motion is essential for investigating particle trajectories on very long times since it satisfies the Liouville theorem. The guiding center motion is known to be governed by the Littlejohn Lagrangian [5]:

$$L(\mathbf{X}, \mu, v_\parallel, \xi) = \frac{e}{c}\mathbf{A} \cdot \dot{\mathbf{X}} + mv_\parallel \frac{(\mathbf{B} \cdot \dot{\mathbf{X}})}{B} + \frac{mc}{e}\mu\dot{\xi} - \mu B - \frac{1}{2}mv_\parallel^2 - e\Phi. \tag{1}$$

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The dynamical variables in this Lagrangian are: $\mathbf{X}$ (the guiding center position), $\mu = mv_\perp^2/2B$ (the magnetic moment), ξ (the gyroangle), and $v_\parallel$ (the parallel velocity). The magnetic field $\mathbf{B}$, the vector potential $\mathbf{A}$, and the scalar potential Φ are evaluated at the guiding center position. The guiding center phase space is six-dimensional, but the six dynamical variables in the Lagrangian do not immediately split into three canonical pairs for Hamiltonian formalism. The reason is that the Littlejohn Lagrangian generally contains time derivatives of four variables rather than three. These four are the components of $\mathbf{X}$ and the gyroangle.

There have been several attempts in the past to eliminate the extra time derivative and find a set of canonical variables (two coordinates and two momenta besides μ and ξ). One of the attempts is based on Boozer coordinates [4], in which the Jacobian is $J = ((\nabla\psi \times \nabla\theta)\cdot\nabla\zeta)^{-1} = 1/B^2$. The contravariant radial component of $\mathbf{B}$ is zero in these coordinates. However, the covariant component B_ψ associated with the nonorthogonality of the coordinates does not generally vanish. The neglect of nonzero B_ψ modifies the Lagrangian and thus the equations of motion, which is undesirable. Later, in Ref. [6], a simple change of the guiding center velocity is introduced to achieve a canonical form with a claim that the orbit in the poloidal plane and the toroidal precession remains unchanged. Yet that procedure is not equivalent to coordinate transformation and does not preserve the Littlejohn Lagrangian either, which can distort the time-dependence of the original gyrocenter motion in long-term simulations.

Reference [7] offers a rigorous alternative. By redefining the poloidal angle, it introduces a new coordinate system in which both the vector potential $\mathbf{A}$ and magnetic field $\mathbf{B}$ only have two nonzero covariant components; this procedure eliminates the $\dot{\psi}$ term from the Lagrangian. Although Ref. [7] gives the poloidal angle transformation analytically for Shafranov equilibrium, it is impractical to find such a coordinate system globally in numerical simulations when the flux surfaces are not circular. The transformation introduced in Ref. [7] gives orthogonal coordinates with one of the coordinate axes being orthogonal to $\mathbf{A}$ and $\mathbf{B}$ locally. It is unfortunate that nonuniformity of this coordinate system is significant in many equilibrium configurations of interest, such as elliptically shaped equilibria. Also, the magnetic field lines are not automatically straight in the orthogonal coordinates. For these reasons, numerical implementation of the orthogonal coordinates is challenging for realistic tokamaks.

To eliminate this difficulty, we herein construct a new type of global coordinates by modifying the poloidal and toroidal angles simultaneously, so that

$$\mathbf{A} = A_\theta(\psi)\nabla\theta + A_\zeta(\psi)\nabla\zeta,$$
$$\mathbf{B} = B_\theta\nabla\theta + B_\zeta\nabla\zeta.$$

It then follows from Eq. (1) that the guiding center Lagrangian can be written as a function of the canonical variables $(\theta, \zeta, P_\theta, P_\zeta)$:

$$L = P_\theta\dot{\theta} + P_\zeta\dot{\zeta} + \frac{mc}{e}\mu\dot{\xi} - H, \tag{2}$$

where the Hamiltonian is defined as

$$H = \mu B + \frac{1}{2}mv_\parallel^2 + e\Phi,$$

with

$$P_\theta = \frac{eA_\theta}{c} + mv_\parallel\frac{B_\theta}{B},$$
$$P_\zeta = \frac{eA_\zeta}{c} + mv_\parallel\frac{B_\zeta}{B},$$

and two of the three coordinates (poloidal and toroidal angles) automatically become canonical variables. These coordinates provide a rigorous Hamiltonian form as in Ref. [7], and they are also suitable globally unlike the orthogonal coordinates. Moreover, the new coordinates preserve the straight magnetic field line feature, which makes them perfectly compatible with existing codes. In what follows, we call them “canonical straight field line coordinates”. In Sec. 2 of this paper, we discuss how to construct such coordinates. Section 3 shows their numerical implementation for realistic tokamaks. Section 4 is a brief summary.

2. Construction of the canonical straight field line coordinates

In the general straight field line coordinates (ψ, θ, ζ), the magnetic field $\mathbf{B}$ has the following contravariant representation:

$$\mathbf{B} = \chi'(\nabla\zeta \times \nabla\psi + q\nabla\psi \times \nabla\theta),$$

and the corresponding covariant representation for the vector potential $\mathbf{A}$ is:

$$\mathbf{A} = \left(\int q\chi'\mathrm{d}\psi\right)\nabla\theta - \chi\nabla\zeta.$$