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Accurate and stable time stepping in ice sheet modeling

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A R T I C L E I N F O

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A B S T R A C T

In this paper we introduce adaptive time step control for simulation of the evolution of ice sheets. The discretization error in the approximations is estimated using "Milne's device" by comparing the result from two different methods in a predictor–corrector pair. Using a predictor–corrector pair the expensive part of the procedure, the solution of the velocity and pressure equations, is performed only once per time step and an estimate of the local error is easily obtained. The stability of the numerical solution is maintained and the accuracy is controlled by keeping the local error below a given threshold using PI-control. Depending on the threshold, the time step Δt is bound by stability requirements or accuracy requirements. Our method takes a shorter Δt than an implicit method but with less work in each time step and the solver is simpler. The method is analyzed theoretically with respect to stability and applied to the simulation of a 2D ice slab and a 3D circular ice sheet. The stability bounds in the experiments are explained by and agree well with the theoretical results.

1. Introduction

There is a growing interest in the prediction of the evolution of the large ice sheets on Antarctica and Greenland and their contribution to the future sea level rise [1–4]. Simulations of the dynamics of ice sheets in the past and in the future have been made, see e.g. [5,6], but improvements in the modeling and the numerical methods are required for better fidelity, accuracy, and efficiency [7]. In this paper, we introduce a method to automatically choose the time steps to control the discretization error and stability of the time integration of the governing system of partial differential equations (PDEs).

The full Stokes (FS) equations for the velocity field in the ice and an advection equation for the evolution of the ice surface are regarded as an accurate model of the motion of glaciers and ice sheets [8–10]. The viscosity in the FS equations depends non-linearly on the velocity. The numerical solution of the equations is therefore demanding in terms of computational time. Hence, different simplifications of the FS equations have been derived under various assumptions to reduce the computing effort. The shallow ice approximation (SIA) is based on the assumption that the thickness of the ice in the vertical direction is small compared to a length scale in the horizontal direction [8]. Other approximations are the shallow shelf approximation (SSA) [11,10] and the Blatter–Pattyn model [12,13]. Comparisons between solutions of the FS equations and the SIA equations are found in [14–16]. The Ice Sheet Coupled Approximation Levels (ISCAL) is an adaptive method using SIA or FS in different parts of the ice sheet [17,18].

Numerical models have been implemented in codes for simulation of large ice sheets. They are often using a finite element method for the FS equations or approximations of them as in [19–23] or a finite volume method as in [24]. The

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PDE to evolve the thickness of the ice is time dependent and in the discretization of the time derivative a time step Δt has to be chosen for accuracy and stability. The stability of a class of one-step schemes with a θ-parameter for the time derivative has been analyzed in [25]. Restrictions on Δt are derived by Fourier analysis of the linearized equations. If Δx is the distance between the nodes in the space discretization then $\Delta t \le C_* \Delta x^2$ for some constant C_*. These one-step schemes are applied to large ice sheets in [26].

The discretization of the PDE in space gives a system of ordinary differential equations (ODEs). In the numerical solution of initial value problems for ODEs, the time step is often chosen to control the estimated local error in the time discretization [27–29]. Given the error estimate and the present time step, a new time step is selected to the next time point such that an error tolerance is satisfied and the solution remains stable [30].

We introduce adaptive time step control (or step size control) for simulation of the ice sheet equations in the community ice sheet model Elmer/Ice [19]. Then the time step varies in the time interval of interest and there is no need to guess a stable and sufficiently accurate Δt for the whole interval in the beginning of the simulation. Spatial derivatives are approximated by the finite element method in Elmer/Ice. The mesh is extruded in the vertical direction from a triangular or quadrilateral mesh in the horizontal plane. It is adjusted in every time step to follow the free boundary at the ice surface. The dominant part of the computational effort is spent on the solution of the equations for the velocity and the pressure in the ice.

The discretization error in the approximations is estimated using "Milne's device" by comparing the result from two different methods in a predictor–corrector pair of Adams type of first and second order accuracy in time [27,31]. The advantage with a predictor–corrector pair is that the expensive part of the procedure, the solution of the velocity and pressure equations, is performed only once per time step and that an estimate of the local error is easily obtained. The time step Δt is chosen to fulfill an error tolerance using PI control according to Söderlind et al. [29,30]. There is a bound on Δt depending on Δx^2 as in [25]. An unconditionally stable method would allow longer Δt but also require a fully implicit method and the solution of several different velocity equations in the iterations to compute the solution in every time step.

The outline of the paper is as follows. The equations that govern the evolution of the ice sheets are stated in Section 2. The predictor method is the Forward Euler method or the second order Adams–Bashforth method and the corrector method is the Backward Euler method or the second order Adams–Moulton method (also referred to as the trapezoidal method) [27] or simplifications of them. The methods are combined in Section 3 to solve for the velocities using FS, SIA, or ISCAL and the advection equation for the thickness. In Section 4, the time step control is introduced. The stability of the methods applied to the thickness equation with the velocity from the SIA equation is analyzed as in [25] in Section 5. In Section 6, the stability of the predictor–corrector scheme is investigated. The time step control is tested in Section 7 by simulation over long time intervals of examples in two and three dimensions from [17,32,33] using the SIA, FS, and ISCAL solvers in Elmer/Ice [17,19]. Conclusions are drawn in the final Section 8.

2. Equations governing the ice sheet dynamics

In this section we describe the equations and solvers for the flow of ice sheets.

2.1. The full Stokes (FS) equations

The flow of an ice sheet can be modeled by the non-linear FS equations [9]. These equations are defined by conservation of mass

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

conservation of momentum

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \nabla \cdot \mathbf{T}^D + \rho \mathbf{g}, \tag{2}$$

and a constitutive equation, the so called Glen's flow law

$$\mathbf{D} = \mathcal{A}(T') f(\sigma) \mathbf{T}^D. \tag{3}$$

Here $\mathbf{v}$ is the vector of velocities $\mathbf{v} = \begin{pmatrix} v_x & v_y & v_z \end{pmatrix}^T$, ρ is the density of the ice and p is the pressure. The deviatoric stress tensor $\mathbf{T}^D$ is given by

$$\mathbf{T}^D = \begin{pmatrix} t^D_{xx} & t^D_{xy} & t^D_{xz} \\ t^D_{yx} & t^D_{yy} & t^D_{yz} \\ t^D_{zx} & t^D_{zy} & t^D_{zz} \end{pmatrix}, \tag{4}$$

where t^D_{xx}, t^D_{yy}, t^D_{zz} and t^D_{xy} denote longitudinal stresses and t^D_{xz}, t^D_{yz} vertical shear stresses. We also have symmetry $t_{xy} = t_{yx}$, $t_{xz} = t_{zx}$ and $t_{yz} = t_{zy}$. The gravitational acceleration in the z-direction is denoted by $\mathbf{g}$, and the total time derivative of the velocity by $\frac{D\mathbf{v}}{Dt}$ which is very small and neglected in glaciological applications. Glen's flow law (3) relates the stress and